CAMBRIDGE MATHEMATICAL SERIES.

KEY TO ELEMENTARY GEOMETRY.

G. BELL AND SONS, LTD.
LONDON: PORTUGAL ST., KINGSWAY
CAMBRIDGE: DEIGHTON, BELL AND CO.
NEW YORK: THE MACMILLAN COMPANY
BOMBAY: A. H. WHEELER AND CO.

A KEY

TO

ELEMENTARY GEOMETRY

BY

W. M. BAKER, M A

AND

A. A. BOURNE, M.A

LONDON

G. BELL AND SONS, LTD.

1917

First published 1904.
Second Edition revised, 1906.
Third Edition revised, 1909.
Fourth Edition revised, 1913.
Fifth Edition revised, 1917.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS
BY ROBERT MACLEHOSE AND CO. LTD.

CONTENTS

KEY TO ELEMENTARY GEOMETRY.

EXERCISES A.

14. Place a straight ruler with its edge against the two end points of the line. Trace a straight line between these points by means of the ruler. If the two lines coincide, the original line is straight.

15. Take any two points A and B on a piece of paper, and place the edge of the ruler against them. Trace a line from A to B along the edge of the ruler. Again place the edge of the ruler against the points A and B, the ruler being *on the other side* of the line AB. Again trace a line from A to B as before. If these two lines coincide, the ruler is straight.

16. Place the straight edge of a ruler on the surface. If in all positions of the ruler the surface touches the edge ot the ruler at every point, the surface is flat.

18. Two planes intersect in a straight line.

EXERCISES B.

1. 3 in. = 7·62 cms. ∴ 1 in. = 25·4 mm.

2. 4 in. = 10·16 cms. nearly ∴ 1 in. = 25·4 mm.

3. 6 cms. = 2·36 in. ∴ 1 cm. = ·39 in. = ·4 in. to the nearest tenth.

4. 8 cms. = 3·15 in. ∴ 1 cm. = ·39 in. = ·4 in. ...

5. 11·45 cms. = $4\frac{1}{2}$ in. nearly.

EXERCISES C.

6. Cut them out and superimpose them.

8. 360, 120, 180, 240.

9. 30, $7\frac{1}{2}$, $22\frac{1}{2}$.

10. 90, 150, 240, 15, 75, $187\frac{1}{2}$°.

14. The angles are rt. ∠s.

17. If the perpendicular is drawn along the edge AB of the set-sq. ABC, turn the set-square over, so that C lies on the other side of AB. Trace another line along the edge AB. If the two lines coincide the perpendicular is a true one.

23. Any two sides together > the third.

30. In an isosceles △.

31. They meet in a pt. This pt. is a pt. of trisection of each.

EXERCISES D.

2. 90°. **3.** 90°. **4.** 90°. **5.** 90°.

EXERCISES E.

2. A regular hexagon. **3.** A square.

EXERCISES G.

3. See Book I., Prop. 31, first method.

EXERCISES H.

3. Make a crease ABC. Fold again so that BC falls along BA. If DBE is the second crease, DBC is a rt. ∠. A fold along DC makes DBC a rt. ∠d. △.

4. Make a crease ABC, and, as in Example 3 above, a second crease DBE at rt. ∠s to it. Fold the double thickness again along DC, and we have an isos. △.

5. Fold so that BC falls along CA, and the crease is at rt. ∠s to ABC at C.

6. Fold the paper about AB. Mark (or prick through) the pt. D where C falls. CD when joined is the reqd. line.

9. Make two creases at rt. ∠s to any st. line AB as in Example 5. These two creases are parallel.

10. and **11.** Use the methods of Examples 5 and 9.

EXERCISES K.

11. By compasses set off an equal length on one of the lines of the paper.

12. 3·6 miles. **13.** 3·6 miles.

14. 19·14 yards. **15.** 8 ft.

16. Let ABCD be the larger sq., CEFG the smaller, G lying in CD, and BCE being a str. line. In CB take a pt. H such that CH = GD. Join FH and cut along the line FH. Also join AH and cut along the line AH. △FEH fitted into ∠FGD, so that FE coincides with FG, and EH falls along GD, and △AHB placed so that AB coincides with AD, and BH is in a str. line with GD, will form a square.

19. On squared paper take AB 4 half inches long, to represent the support ($\frac{1}{2}$ in. represents 1 foot). Along the line thro. B take BC 3 half inches long. $AC^2 = 3^2 + 4^2$ (Exercises D. 2), and AC = 5 half in. Produce CA to D, making AD = AC. Read off the perp. distance of D from CB produced. This is equal to 8 half inches ∴ the end of the see-saw can rise 8 ft. from the ground. Also by measurement the ∠ reqd. = ∠ACB = 53°

21. On squared paper take AC 24 units long, and AB at rt. ∠s to it 10 units long. BC represents the rope. With centre B and rad. BC describe a circle meeting BA produced at D. Read off BD and we find it to be 26 units ∴ the rope is 26 feet long.

22. Take AB 2·4 in long and BC at rt. ∠s to it ·7 in. long. With centre A and rad. AC describe a circle cutting AB produced at D. Reading off AD, we find it to be 2·5. ∴ the length reqd. = 25 feet.

23. The angle is 90°; length of crease $3\frac{3}{4}$ inches. See Ex. P. 12.

EXERCISES P.

4. 6 ft. 8ft.

9. See Ex. XXIII. 4.

12. If EOF is the crease, E lying in CD, O in AC, F in AB, EOC is a rt. $\angle$. $AC^2 = AD^2 + DC^2 = 3^2 + 4^2 = 5^2$ $\therefore$ $AC = 5$ ft. $\triangle EOC$ is equiangular to $\triangle ADC$ $\therefore$ $\frac{EO}{OC} = \frac{AD}{DC}$ *i.e.* $\frac{EO}{\frac{5}{2}} = \frac{3}{4}$ $\therefore$ EO $= \frac{15}{8}$ ft. $\therefore$ the crease $= \frac{15}{4} = 3\frac{3}{4}$ ft.

EXERCISES I.

1. The pencil mark has *some* width and is therefore not a true line.

2. The dot has *some* length and breadth and is therefore not a true point.

18. Two planes intersect in a str. line.

EXERCISES II.

1. The ext. $\angle$s are respectively supplementary to the int. and equal $\angle$s (I. 1.) and are therefore equal.

2. Similar to the above.

3. If BE, BF are the bisectors, $\angle EBF = \angle EBA + \angle FBA = \frac{1}{2}(\angle ABC + \angle ABD) =$ a rt. $\angle$.

4. If the crease DE meets AB at C, the adj. $\angle$s ACD, BCD coincide when we fold, and are therefore equal. They are also adj. $\angle$s and are therefore rt. $\angle$s.

5. The four $\angle$s at the meeting pt. are equal to four rt. $\angle$s $\therefore$ two adj. $\angle$s $= 2$ rt. $\angle$s. The theorem thus follows by I. 2.

6. $\angle AOP + \angle AOQ = \angle BOQ + \angle AOQ = 2$ rt. $\angle$s (I. 1.) $\therefore$ POQ is a str. line (I. 2.).

7. Any two adj. $\angle$s $=$ half all the $\angle$s at the meeting pt. $= 2$ rt. $\angle$s. The proposition thus follows by I. 2.

8. Let EO be the bisector of $\angle AOD$, and OF the bisector of the opp. vert. $\angle BOC$. $\angle AOE + \angle AOF = \angle AOE + \angle AOC + \angle COF = \angle AOE + \angle AOC + \frac{1}{2}\angle COB = \angle AOE + \angle AOC + \frac{1}{2}\angle AOD = \angle AOD + \angle AOC = 2$ rt. $\angle$s $\therefore$ EOF is a str. line (I. 2.).

9. Produce AB, AC, sides of $\triangle ABC$ to D and E. $\angle$s ABC, DBC $= 2$ rt. $\angle$s (I. 1.) $\angle$ACB, ECB $= 2$ rt. $\angle$s (I. 1.) $\therefore$ these four $\angle$s $= 4$ rt. $\angle$s.

10. They are in a str. line by I. 2.

11. They are in a str. line, as in Example 6, above.

EXERCISES III.

1. $\triangle$s ABC, AED are equal in all respects by I. 4.

2. $\triangle$s ABC, DEF are equal in all respects by I. 4.

3. $\triangle$s ACD, BCD are equal in all respects by I. 4.

4. As in Example 2 above $AC = AE$ and $AE = EC$.

5. $\angle AEC = \angle BED$ (I. 3.) $\therefore$ $\triangle$s AEC, BED are equal in all respects by I. 4.

6. This follows at once from I. 4.

7. Join AC, XZ. From $\triangle$s ABC, XYZ, $AC = XZ$ and $\angle ACB = \angle XZY$ (I. 4.). From $\triangle$s ACD, XZW, $AD = WX$ (I. 4.). Also by the same prop. the $\angle$s are equal respectively $\therefore$ the quad[ls] are equal in all respects.

8. Let E be the mid. pt. of CD, Join AE, BE From $\triangle$s ACE, BDE, $AE = BE$ (I. 4.).

9. This follows at once from $\triangle$s ABE, ACD (I. 4.).

10. $\triangle ABF = \triangle DAE$ (I. 4.). Take $\triangle AGE$ from each and fig. $GEBF = \triangle AGD$.

11. This follows from $\triangle$s ACD, BCD by I. 4.

12. This follows from I. 4.

EXERCISES IV.

1. In $\triangle$s ABF, ACF, $AB = AC$; AF is common; and $\angle BAF = \angle CAF$ $\therefore$ $\angle ABF = \angle ACF$ (I. 4.) $\therefore$ also their supplements $\angle$s DBC, ECB are equal (I. 1.).

2. In $\triangle$s BAO, CAO $\angle ABO = \angle ACO$ (I. 4.) $= \frac{1}{2} \angle ACB = \frac{1}{2} \angle ABC$.

3. Let D, E, F be the mid. pts. of AB, AC, BC respectively. By I. 5 $\angle DBF = \angle ECF$. $\therefore$ from $\triangle$s DBF, ECF, $DF = EF$ (I. 4.).

4. From $\triangle$s BAQ, CAP, $BQ = CP$ (I. 4.).

5. From $\triangle$s BAQ, CAP, $BQ = CP$ (I. 4.).

6. From $\triangle$s ACP, ABQ, $\angle ACP = \angle ABQ$ (I. 4.). Also their parts $\angle$s ACB, ABC are equal (I. 5.) $\therefore$ $\angle BCP = \angle CBQ$.

7. From $\triangle$s AFE, BDF, $EF = DF$ (I. 4.). In the same way, $DE = EF$ $\therefore$ $\triangle DEF$ is equilateral.

8. $\angle ABD = \angle ACE$ (I. 5.) $\therefore$ from $\triangle$s ABD, ACE, $AD = AE$ (I. 4.).

9. Produce AD to E. In $\triangle$s BAD, CAD, $\angle$BDA $=$ $\angle$CDA (I. 4.). $\therefore$ their supplements $\angle$s BDE, CDE are equal. For proof that CD bisects $\angle$ACB, see Exercise 2 above.

10. Each of the angles $= 66\frac{1}{2}°$. They are equal by I. 5.

EXERCISES V.

1. Let the sides AB, AC of $\triangle$ABC be produced to D and E, so that $\angle$DBC $=$ $\angle$ECB. Their supplements $\angle$s ABC, ACB are equal. $\therefore$ AB $=$ AC and the $\triangle$ABC is isosceles (I. 6.).

2. This follows at once from I. 6.

3. OB $=$ OC $\therefore$ $\angle$OBC $=$ $\angle$OCB (I. 5.) $\therefore$ 2$\angle$OBC $=$ 2$\angle$OCB *i.e.* $\angle$ABC $=$ $\angle$ACB $\therefore$ AB $=$ AC (I. 6.).

4. Produce DA, CB to meet in O. OD $=$ OC (I. 6.). $\angle$DAB $=$ $\angle$CBA $\therefore$ their supplements $\angle$s OAB, OBA are equal (I. 1.). $\therefore$ OA $=$ OB (I. 6.). But OD $=$ OC $\therefore$ AD $=$ BC.

5. In $\triangle$s DCB, EBC, $\angle$DBC $=$ $\angle$ECB (I. 4.) $\therefore$ BA $=$ AC (I. 6.).

6. $\angle$DBC $>$ $\angle$ABC. But $\angle$ABC $=$ $\angle$ACB (I. 5.) $\therefore$ $\angle$DBC $>$ $\angle$DCB.

7. From $\triangle$s AEB, CDE, EB $=$ EC (I. 4.).

8. Let the sides AB, AC in $\triangle$ABC be unequal. If $\angle$ABC $=$ $\angle$ACB, then AB $=$ AC (I. 6.) which is contrary to the hypothesis.

9. Let the angles ABC, ACB of $\triangle$ABC be unequal. If AB $=$ AC, $\angle$ABC $=$ $\angle$ACB, which is contrary to the hypothesis.

10. CA should be equal to CB, by I. 6.

11. The exterior $\angle$s should be 136° and 132° by I. 1. The third $\angle$ of the $\triangle$ will be found to be 88°.

EXERCISES VI.

1. This follows by I. 7 from $\triangle$s ACB, ADB.

2. From $\triangle$s ACB, ADB, $\angle$CAE $=$ $\angle$DAE (I. 7.) $\therefore$ the $\triangle$s CAE, DAE are equal in all respects by I. 4.

3. Let EO, FO bisect AD and BC at rt. $\angle$s at E and F. From $\triangle$s AEO, DEO, AO $=$ DO (I. 4.). From $\triangle$s BFO, CFO, BO $=$ CO (I. 4.). $\therefore$ $\triangle$s AOB, COD are equal in all respects (I. 7.)

4. $\angle$GBC $=$ $\angle$GCB $\therefore$ GB $=$ GC (I. 6.) $\therefore$ from $\triangle$s AGB, AGC $\angle$BAG $=$ $\angle$CAG (I. 7.).

5. Take O the centre of the circle, and let AC meet BO at E. From $\triangle$s AOB, COB, $\angle$AOB $=$ $\angle$CBO (I. 7.). Then from $\triangle$s ABE, CBE, AE $=$ EC, and $\angle$AEB $=$ $\angle$CEB (I. 4.). Also these $\angle$s are adj. and therefore rt. $\angle$s.

6. Let ADB be the new position of the $\triangle$ABC. Join CD, meeting AB at E. From $\triangle$s ACB, ADB, $\angle$ABC $=$ $\angle$ABD, for one is a copy of the other. From $\triangle$s BEC, BED, CE $=$ DE and $\angle$CEB $=$ $\angle$DEB (I. 4.). Also these $\angle$s are adj. and therefore rt. $\angle$s.

7. Join EB. From $\triangle$s EDB, BCE, $\angle$EDB $=$ $\angle$BCE (I. 7.).

8. From $\triangle$s ACB, ADB, $\angle$CBA $=$ $\angle$DAB, and $\angle$CAB $=$ $\angle$DBA (I. 7.) $\therefore$ $\angle$DAC $=$ $\angle$DBC. Also AO $=$ BO (I. 6.) $\therefore$ $\triangle$s AOC, BOD are equal in all respects (I. 4.).

9. Take a pt. P equidistant from A and D. Join OP. From $\triangle$s AOP, DOP $\angle$AOP $=$ $\angle$DOP (I. 7.). But $\angle$AOB $=$ $\angle$COD (hyp.) $\therefore$ $\angle$POB $=$ $\angle$POC $\therefore$ from $\triangle$s POB, POC, PB $=$ PC (I. 4.).

10. Let AC, DB meet at O. $\angle$ADB $=$ ABD (I. 5.) $\therefore$ $\angle$CDB $=$ $\angle$CBD $\therefore$ CD $=$ CB (I. 6.) $\therefore$ from $\angle$s ADC, ABC, $\angle$DAO $=$ $\angle$BAO (I. 7.) $\therefore$ from $\angle$s AOD, AOB, DO $=$ OB and $\angle$AOD $=$ $\angle$AOB $=$ a rt. $\angle$ (I. 4.).

11. From $\triangle$s DAC, EBC, $\angle$DAC $=$ $\angle$EBC (I. 7.) $\therefore$ from $\triangle$s DAB, EBA, DB $=$ EA (I. 4.).

12. From $\triangle$s ABC, CDA, $\angle$DAC $=$ $\angle$BCA (I. 7.) $\therefore$ $\angle$OAC $=$ $\angle$OCA $\therefore$ OA $=$ OC (I. 6.). In the same way from $\triangle$s DAB, BCD, OB $=$ OD.

13. From $\triangle$s AOD, COD, $\angle$AOD $=$ $\angle$COD (I. 7.) from $\triangle$s AOB, COB, $\angle$AOB $=$ $\angle$COB (I. 7.) $\therefore$ $\angle$AOD $+$ $\angle$AOB $=$ $\frac{1}{2}$ the 4 $\angle$s round O $=$ 2 rt. $\angle$s $\therefore$ DOB is a str. line (I. 2.).

14. From $\triangle$s AOE, COE, $\angle$AOE $=$ $\angle$COE (I. 7) $=$ $\frac{1}{2}$$\angle$AOC $=$ $\frac{1}{2}$$\angle$BOD (I. 3.). From $\triangle$s DOF, BOF, $\angle$DOF $=$ $\angle$BOF $=$ $\frac{1}{2}$$\angle$BOD $\therefore$ $\angle$DOF $=$ $\angle$COE. Add $\angle$EOD to each and $\angle$DOF $+$ $\angle$DOE $=$ $\angle$COE $+$ $\angle$DOE $=$ 2 rt. $\angle$s (I. 1.) $\therefore$ EOF is a str. line (I. 2.).

15. OA $=$ OB $\therefore$ $\angle$OAB $=$ $\angle$OBA (I. 5.) $\angle$EOA $=$ $\angle$DOB (I. 3.) $\therefore$ from $\triangle$s EOA, DOB, $\angle$EAO $=$ $\angle$DBO (I. 4.) $\therefore$ $\angle$EAB $=$ $\angle$DBA $\therefore$ AC $=$ BC (I. 6.).

16. BF = CF (I. 6.) ∠EFB = ∠DFC (I. 3.). But ∠AFE = ∠AFD ∴ ∠AFB = ∠AFC ∴ from △s AFB, AFC, AB = AC (I. 4.).

17. From △s ADB, ADC, ∠BDA = ∠CDA (I. 4.) = a rt. ∠.

18. From △s AEO, BEO, ∠AOE = ∠BOE (I. 7.) ∴ ∠DOF = ∠COF (I. 3.) ∴ from △s DOF, COF, DF = CF (I. 4.).

19. From △s DOA, DOB, ∠DOA = ∠DOB (I. 7.) = a rt. ∠. We have drawn a perp. to AB at its mid. pt. O.

EXERCISES VII.

1. If possible let there be drawn from the point A to the str. line BC three equal str. lines AB, AC, AD. AD = AC ∴ ∠ADC = ∠ACD (I. 5.) AB = AC ∴ ∠ABC = ∠ACB (I. 5.) ∴ ext. ∠ABC = int. opp. ∠ADB which is impossible (I. 8.).

2. If a circle whose centre is A cut a str. line at three points, B, C, D, we should have three equal str. lines from a pt. to a str. line, which is impossible by the preceding.

3. If in △ABC, ∠ABC = ∠ACB = a rt. ∠, produce BC to D. ∠ACD = a rt. ∠ = int. opp. ∠ABC which is impossible (I. 8.).

4. This follows at once from I. 4.

5. As in the preceding △AEB = △CEF. Adding △BEC to each, △ABC = △BCF.

6. ∠CBD = supplement of ∠CBA = 116° ∴ ∠CBD is gr. than ∠CAB by 77°.

7. From △s BOD, AOD, ∠BDO = ∠ADO (I. 4.) = a rt. ∠.

8. ∠DOC = ∠AOB (I. 3.) ∴ from △s DOC, BOA, CD = AB = 8 cms. (I. 4.).

9. With centre B and rad. BC describe a circle to meet BA produced at D. Read off the length of BD, estimating the second decimal place (BD = 5·83 cms.).

EXERCISES VIII.

1. Bisect BC at H. Join AH and produce it to K, making HK equal to AH. Join CK. ∠KHC = ∠AHB (I. 3.) ∴ from △s KHC, AHB, ∠KCH = ∠ABH (I. 4.) ∴ ∠HCG *i.e.* ∠ACD is gr. than ∠ABC.

2. Produce DA to E, BC to F. Join DB. Ext. $\angle$BAE $>$ int. opp. $\angle$ABD (I. 8.) Also ext. $\angle$DCF $>$ int. opp. $\angle$DBC (I. 8.) $\therefore$ sum of the $\angle$s BAE, DCF is gr. than $\angle$ABC. In the same way the sum of the $\angle$s BAE, DCF is gr. than $\angle$ADC.

3. Let CO produced meet DB at E. Ext. $\angle$COD $>$ int. opp. $\angle$DEO of $\triangle$DEO (I. 8.). Ext. $\angle$DEO $>$ int. opp. $\angle$EBC of $\triangle$EBC (I. 8.) $\therefore$ $\angle$COD $>$ $\angle$DBC. Also in $\triangle$DAB 2 rt. $\angle$s $>$ the two $\angle$s DAB, DBA $\therefore$ $\angle$COD + 2 rt. $\angle$s $>$ $\angle$DBC + $\angle$DAB + $\angle$DBA *i.e.* $>$ $\angle$DAB + $\angle$CBA $\therefore$ $\angle$DAB + $\angle$CBA differ from $\angle$COD by less than 2 rt. $\angle$s.

4. Take a quad[l] ABCD. Place a pencil on the line AB, pointing from A to B. Turn the pencil about the pt. A through the $\angle$BAD, the pencil then pointing from A to D. Next turn the pencil about D to the position CD, the pencil now pointing from C to D. Again turn the pencil about C into the position CB, the pencil now pointing from C to B. To complete a turn through 4 rt. $\angle$s we shall have to turn the pencil further about B through the angle CBA $\therefore$ the three $\angle$s BAD, ADC, DCB are less than 4 rt. $\angle$s.

5. Join A to any pt. D in BC. Ext. $\angle$ADC $>$ int. opp. $\angle$ABC (I. 8.). Ext. $\angle$BDA $>$ int. opp. $\angle$ACB (I. 8.) $\therefore$ the two $\angle$s BDA, CDA are together $>$ $\angle$s ABC, ACB, *i.e.* $\angle$s ABC, ACB are together $<$ 2 rt. $\angle$s.

6. Two obtuse angles are together $>$ 2 rt. $\angle$s $\therefore$ no $\triangle$ can have two obtuse angles (I. 9.).

7. On sqd. paper take AB, a vertical line, 4 in. long to represent 48 ft. With centre A and rad. 5 in. (to represent 60 ft.) describe a circle cutting the horizontal line thro. B at C. Read off BC ($=3$ in.) and we find the dist. reqd. is 36 ft.

8. ABD will be a rt. angle.

9. The $\angle$s will be rt. $\angle$s.

10. By folding as in Exercises C. 15, make a rt. $\angle$DEF. Fold so that EF falls along ED. The new crease will bisect $\angle$DEF, and $\therefore$ make $\angle$s of 45° with ED and EF.

11. Repeat Exercise H. 3.

EXERCISES IX.

1. Let ABC be a $\triangle$, rt. $\angle$d. at B. Produce AB to D. $\angle$ABC $=\angle$CBD$>$int. opp. $\angle$BCA (I. 8.) $\therefore$ AC$>$AB. Similarly AC$>$BC.

2. Let ABC be a $\triangle$ having AB less than AC $\therefore$ $\angle$ACB is less than $\angle$ABC. If $\angle$ACB were obtuse or a rt. $\angle$, $\angle$ACB$+\angle$ABC would be gr. than two rt. $\angle$s, which is impossible (I. 9.).

3. The greatest $\angle$ is a rt. $\angle$.

4. If x and y are the other sides of the reqd. $\triangle$, $\frac{x}{44}=\frac{y}{54}=\frac{90}{60}$ $\therefore$ $x=66$ mms. and $y=81$ mms.

5. BC on paper$=5{\cdot}05$ cms. approx. $\therefore$ the dist. reqd.$=$ 253 yds. nearly.

6. Fold the paper so that BD falls on BC. If the crease coincides with BA, BA is perp. to CD.

EXERCISES X.

1. Take any quad[l] ABCD. Join AC. AB$+$BC$>$AC (I. 12.) $\therefore$ AB$+$BC$+$CD$>$AC$+$CD$>$AD (I. 12.).

2. AD$+$AC$>$DC (I. 12.) $\therefore$ BD$+$BC$+$AD$+$AC$>$BD$+$BC $+$DC *i.e.* perimeter of $\triangle$ABC$>$perimeter of $\triangle$BDC.

3. Let PQ, SR meet in O. Then OQ$=$OR (I. 6.). But $\angle$RSP is gr. than, equal to, or less than $\angle$SPQ according as OP is gr. than, equal to, or less than OS *i.e.* as PQ is gr. than, equal to, or less than RS.

4. Ext. $\angle$EDC$>$int. opp. $\angle$ABC (I. 8.)$>\angle$ACB for $\angle$ABC $=\angle$ACB (I. 5.)$>\angle$ECD $\therefore$ EC$>$ED (I. 11.) $\therefore$ EC$>$EA.

5. See Exercises VII., Example 1.

6. Let ABCD be the quadl. AB$+$BC$>$AC (I. 12.) and AD $+$DC$>$AC (I. 12.) $\therefore$ AB$+$BC$+$CD$+$DA$>$2AC. Similarly AB$+$BC$+$CD$+$DA$>$2BD $\therefore$ adding and dividing by 2, AB$+$BC$+$CD$+$DA$>$AC$+$BD.

7. Let the diagonals of the quadl. ABCD cut at the pt. O. AO$+$OB$>$AB, BO$+$OC$>$BC, CO$+$OD$>$CD, DO$+$AO$>$AD (I. 12.) $\therefore$ adding and dividing by 2, AC$+$DB$>\frac{1}{2}$(AB$+$BC$+$CD $+$AD).

8. Produce AO to meet BC at D. $\angle$ADB$>$int. opp. $\angle$ACB (I. 8.)$>\angle$ABD $\therefore$ AB$>$AD$>$AO. Also BO$+$OC$>$BC (I. 12.) $>$AB$>$AO.

9. Take any pt. O within the quadl. and not at their intersection. BO$+$OD$>$BD (I 12.). AO$+$OC$>$AC (I. 12.) $\therefore$ AO$+$BO$+$CO$+$DO$>$AC$+$BD.

10. Take O the common centre, A a pt. on the larger circle. Let OA meet the smaller circle at B. We have to prove that AB is shorter than any other line drawn from A to the smaller circle. Take any pt. C on the smaller circle. Join OC, CA. OC$+$CA$>$OA (I. 12.) *i.e.* OC$+$CA$>$OB$+$BA $\therefore$ CA$>$BA.

11. Let ABCDEF be any rectil. fig. O any point within it. Join OA, OB, etc. OA$+$OB$>$AB, OB$+$OC$>$BC, and so on. $\therefore$ adding, 2(OA$+$OB$+$OC$+$OD$+$OE$+$OF)$>$the perimeter. $\therefore$ OA$+$OB$+$etc.$>\frac{1}{2}$ perimeter.

12. $\angle$BAC is obtuse $\therefore$ BD$>$BA and CE$>$CA (I. 11.) $\therefore$ BD $+$CE$>$BA$+$CA$>$BE$+$CD$+$EA$+$DA, but EA$+$DA$>$ED (I. 12.) $\therefore$ BD$+$CE$>$BE$+$CD$+$DE.

13. Produce AD to G, and make DG equal to DA. $\angle$ADC$=$ opp. vert. $\angle$BDG (I. 3.) $\therefore$ from $\triangle$s BDG, CDA, BG$=$AC (I. 4.) Also AB$+$BG$>$AG (I. 12.) *i.e.* AB$+$AC$>$2AD. Similarly AC$+$ CB$>$2CF and CB$+$BA$>$2BE $\therefore$ adding AB$+$BC$+$CA$>$AD $+$BE$+$CF.

14. AD$+$DB$>$AB (I. 12.) *i.e.* 2AD$+$BC$>$2AB. Similarly, 2BE$+$AC$>$2BC, 2CF$+$AB$>$2CA $\therefore$ adding 2(AD$+$BE$+$CF)$>$ AB$+$BC$+$CA.

15. Thro. C draw ACD equal to the string. With centre A and rad. AB desc. a circle, and with centre C and rad. CD desc. a second circle meeting the first at B and E. CB$=$CD radii $\therefore$ AC$+$CB$=$the string $\therefore$ AB is the reqd. position of the rod. The position AE gives a second solution. For a solution to be possible, the circles must meet. AC$+$CB must be$>$AB, *i.e.* the string must be longer than the rod.

EXERCISES XI.

1. BA$<$BD$+$DA (I. 12.) CA$<$CD$+$DA (I. 12.) $\therefore$ BA$+$CA$<$ BD$+$CD$+$2DA *i.e.* the diff. of BA$+$CA and BD$+$CD$<$2DA.

2. Take any pt. O within the $\triangle$ ABC. From $\triangle$s DAC, BAC, $\angle$DAC $=$ $\angle$BAC (I. 7.) $\therefore$ $\angle$DAO $>$ BAO $\therefore$ from $\triangle$s DAO, BAO, DO $>$ BO (I. 14.).

3. Let D be any pt. on the bisector of the $\angle$BAC, and let DE, DF be drawn perp. to AB and AC respectively. From $\triangle$s AED, AFD, DE $=$ DF (I. 16.).

4. Let BD, CE be drawn from the extremities of the base BC of the isos. $\triangle$ ABC perp. to the opp. sides. $\angle$ABC $=$ $\angle$ACB (I. 5.) $\therefore$ from $\triangle$s EBC, DCB, CE $=$ DB (I. 16.).

5. Let the diagonal AC of the quadl. ABCD bisect the $\angle$s at A and C, and let BD meet AC at O. From $\triangle$s ADC, ABC, CD $=$ CB (I. 16.) $\therefore$ from $\triangle$s DOC, BOC, DO $=$ OB and $\angle$DOC $=$ $\angle$BOC $=$ a rt. $\angle$ (I. 4.).

6. Let AD, BE meet at P, BE and CF at Q, CF and AD at R. The angles of $\triangle$ ABC are all equal (I. 5.) $\therefore$ $\triangle$s ADC, BEA CFB are equal in all respects (I. 4.) $\therefore$ $\triangle$s RDC, QFB, PEA are equal in all respects (I. 16.) $\therefore$ $\angle$CRD $=$ $\angle$APE $=$ $\angle$ BQF $\therefore$ $\angle$PRQ $=$ $\angle$RPQ $=$ $\angle$PQR (I. 3.) $\therefore$ PQ $=$ QR $=$ RP (I. 6.).

7. From $\triangle$s BAD, ABC, $\angle$ABD $=$ $\angle$BAC (I. 4,). Also $\angle$AEB $=$ $\angle$BFA (hyp.) $\therefore$ $\triangle$s AEB, BFA are equal in all respects (I. 16.).

EXERCISES XII.

1. $\angle$FAD $=$ $\angle$DAE (Hyp.) $=$ alt $\angle$FDA (I. 20.) $\therefore$ FA $=$ FD (I 6.). Then from $\triangle$s DFA, DEA, DE $=$ DF and EA $=$ FA (I. 16.) $\therefore$ AEDF is equilateral.

2. Let PN be the perp. on the line thro. A, PM the perp. on that thro. B. $\angle$PAN $=$ alt. $\angle$PBM (I. 20.) $\therefore$ from $\triangle$s PAN, PBM, PA $=$ PB (I. 16.).

3. Let FAE be $\parallel$ to BC, DBF $\parallel$ to CA, and ECD $\parallel$ to AB. $\angle$ABC $=$ alt. $\angle$BCD (I. 20.) $=$ int. opp. $\angle$DEF (I. 20.) Similarly, $\angle$BAC $=$ $\angle$ FDE and $\angle$ ACB $=$ $\angle$ DFE.

4. Let $\triangle$s ABC, DBC be on the same base BC, and between the same $\parallel$s AD and BC such that DC bisects AB at O. $\angle$ OAD $=$ alt. $\angle$OBC (I 20.) and $\angle$DOA $=$ $\angle$COB (I. 3.) $\therefore$ DO $=$ OC (I. 16.). Also $\angle$BOD $=$ $\angle$COA (I. 3.) $\therefore$ from $\triangle$s BOD, AOC, $\angle$BDO $=$ $\angle$ACO (I. 4.) $\therefore$ BD is $\parallel$ to CA.

5. $\frac{1}{2}\angle$ AED $>\frac{1}{2}\angle$ BAC (I. 8.) *i.e.* $\angle$ ACF $>$ alt. $\angle$ CAE $\therefore$ CF is not $\parallel$ to EA.

6. $\angle$CDE = int. opp. $\angle$CAB (I. 20.) = $\angle$ABC (I. 5.) = ext. $\angle$DEC (I. 20.) $\therefore$ CD = CE (I. 6.) $\therefore$ DA = EB $\therefore$ from $\triangle$s DAB, EBA, $\angle$DBA = $\angle$EAB (I. 4.). Also $\angle$EDF = alt. $\angle$DBA (I. 20.) = $\angle$EAB = alt. $\angle$FED (I. 20.) $\therefore$ DF = EF (I. 6.).

7. Let O be the centre of the circle. Join OC, OD. $\angle$OCA = $\angle$OAC (I. 5.) = alt. $\angle$OBD (I. 20.) = $\angle$ODB (I. 5.) $\therefore$ from $\triangle$s AOC, DOB, AC = BD (I. 16.).

8. Let O be the centre of the circle, and EBF be drawn $\parallel$ to CD. $\angle$EBC = alt. $\angle$BCO (I. 20.) = $\angle$CBO (I. 5.). Similarly $\angle$FBD = $\angle$OBD.

9. Let ABCD be a quadl. having BC $\parallel$ to AB but not equal to it. If possible let AB be $\parallel$ to CD. Then $\angle$BAC = alt. $\angle$ACD (I. 20.) and $\angle$BCA = alt. $\angle$CAD (I. 20.) $\therefore$ from $\triangle$s ABC, CDA, BC = AD (I. 16.). Which is contrary to the hypothesis.

10. Let AF the perp. at A meet OB at E and let BG be the perp. at B. If possible let AF be $\parallel$ to BG. Then $\angle$EBG = ext. $\angle$OEF (I. 20.) wh. $>$ int. opp. $\angle$OAE (I. 8.) but $\angle$EBG = $\angle$OAE by hyp. $\therefore$ AF and BG cannot be $\parallel$ *i.e.* they must meet if produced.

11. From $\triangle$s DBC, ECB, DB = EC (I. 16.) $\therefore$ AD = AE. If DE is not $\parallel$ to BC let DF $\parallel$ to BC meet AC at F. $\angle$ADF = int. opp. $\angle$ABC (I. 20.) = $\angle$ACB (I. 5.) = ext. $\angle$AFD (I. 20.) $\therefore$ AF = AD = AE wh. is absurd $\therefore$ DE must be $\parallel$ to BC. Similarly, MN is $\parallel$ to BC $\therefore$ MN is $\parallel$ to DE.

12. Draw AM perp. to BC, and EH $\parallel$ to BC to meet AM at H. Draw DK $\parallel$ to AH to meet EH in K. From $\triangle$s ADM, DEK, AM = DK (I. 16.). From $\triangle$s DKH, HMD, DK = MH (I. 20. and 16.) $\therefore$ MH = AM $\therefore$ H is a fixed point and EH is a fixed line.

EXERCISES XIII.

1. Let n be the number of sides. 8 rt. $\angle$s + 4 rt. $\angle$s = $2n$ rt. $\angle$s (I. 22. Cor. 1.) $\therefore n = 6$.

2. Ext. $\angle = 30^\circ$ $\therefore 30n = 360$ (I. 22. Cor. 2.) $\therefore n = 12$.

3. Let x be the number of rt. $\angle$s in each angle. (1) $8x + 4 = 16$ (I. 22. Cor. 1.) $\therefore x = 1\frac{1}{2}$. (2) $10x + 4 = 20$ (I. 22. Cor. 1.) $\therefore x = \frac{8}{5} = 1\frac{3}{5}$. (3) $7x + 4 = 14$ (I. 22. Cor. 1.) $\therefore x = \frac{10}{7} = 1\frac{3}{7}$.

4. Int. $\angle$s $= 4$ rt. $\angle$s (I. 22. Cor. 2.) $\therefore$ if n be the number of sides, $4 + 4 = 2n$ (I. 22. Cor. 1.) $\therefore n = 4$.

5. Let the smallest $\angle$ be equal to x rt. $\angle$s. Then $(1 + 3 + 6 + 9 + 11)x + 4 = 10$ $\therefore x = \frac{6}{30} = \frac{1}{5}$ $\therefore$ the angles are respectively $\frac{1}{5}, \frac{3}{5}, \frac{6}{5}, \frac{9}{5}, \frac{11}{5}$ of a rt. $\angle$.

6. If n and $2n$ be the numbers of sides of the polygons, and α and β rt. $\angle$s in an angle of each, $n\alpha + 4 = 2n$ (I. 22. Cor. 1.). $2n\beta + 4 = 4n$ (I. 22. Cor. 1.) $\therefore \alpha = \dfrac{2n-4}{n}$ and $\beta = \dfrac{2n-2}{n}$

$\therefore \dfrac{\frac{2n-4}{n}}{\frac{2n-2}{n}} = \dfrac{8}{9}$, whence $n = 10$, and the polygons have 10 and 20 sides respectively.

7. Let each angle be equal to x rt. $\angle$s. $5x + 4 = 10$ (I. 22. Cor. 1.) $\therefore x = \frac{6}{5}$. $\therefore$ $\angle$ABE $+ \angle$AEB $= \frac{4}{5}$ of a rt. $\angle$ (I. 22.) *i.e.* $2\angle$ABE $= \frac{4}{5}$ of a rt. $\angle$ (I. 5.). $\angle$ABC $- \frac{1}{2}\angle$ABE $= (\frac{6}{5} - \frac{1}{5})$ of a rt. $\angle =$ a rt. $\angle$.

8. Let the bisectors of the $\angle$s A and B of the quadl. ABCD meet at O. $\angle$BAO $+ \angle$ABO $= \frac{1}{2}(\angle$DAB $+ \angle$ABC$) =$ a rt. $\angle$ (I. 20) $\therefore$ $\angle$AOB $=$ a rt. $\angle$ (I. 22.).

9. BD $>$ AD $\therefore$ $\angle$BAD $>$ $\angle$DBA (I. 10.). CD $>$ AD $\therefore$ $\angle$CAD $>$ $\angle$DCA $\therefore$ adding $\angle$BAC $>$ $\angle$ABC $+ \angle$ACB $\therefore$ $\angle$BAC is obtuse (I. 22.).

10. Not more than one $\angle$ of a $\triangle$ can be obtuse (I. 22.) $\therefore$ since the ext. $\angle$s are supplements of the adj. int. $\angle$s, two, at least, of the ext. $\angle$s must be obtuse.

11. This follows at once from the fact that the three $\angle$s of a $\triangle$ are equal to two rt. $\angle$s (I. 22.).

12. Draw a diagonal dividing the fig. into two $\triangle$s. The propⁿ follows at once from (I. 22.).

13. Two of the angles together are equal to a rt. $\angle$ $\therefore$ the third $\angle$ is a rt. $\angle$ (I. 22.).

14. The greatest $\angle = \frac{1}{2}$ the sum of all three $\angle$s $=$ a rt. $\angle$ (I. 22).

15. The greatest $\angle$ of the $\triangle$ $>$ half all three $\angle$s and is $\therefore$ gr. than a rt. $\angle$ (I. 22.).

16. Let the isos. $\triangle$s ABC, DEF have their vertical $\angle$s at A and D equal. $\angle$ABC + $\angle$ACB = supplement of $\angle$A (I. 22.) $\therefore$ $\angle$ACB = $\frac{1}{2}$ supplement of $\angle$A (I. 5.). Similarly, $\angle$DEF = $\frac{1}{2}$ supplement of $\angle$D = $\frac{1}{2}$ supplement of $\angle$A $\therefore$ $\angle$DEF = $\angle$ACB $\therefore$ the $\triangle$s are equiangular (I. 22.).

17. Let AD be drawn to the mid. pt. D of the base of the $\triangle$ABC. Produce AD to E making DE equal to DA. Join BE. $\angle$BDE = $\angle$CDA (I. 3.) $\therefore$ from $\triangle$s BDE, CDA, BE = AC and $\angle$BED = $\angle$DAC (I. 4.) $\therefore$ BE is $\parallel$ to AC (I. 18.) $\therefore$ $\angle$BAC + $\angle$ABE = 2 rt. $\angle$s (I. 20.). Hence, if $\angle$BAC < a rt. $\angle$, $\angle$ABE > a rt. $\angle$ $\therefore$ from $\triangle$s ABE, BAC, AE > BC (I. 14.) *i.e.* AD > half the base BC. Similarly, if $\angle$BAC > a rt. $\angle$, AD < half the base BC, and if $\angle$BAC = a rt. $\angle$, AD = half the base BC (I. 4.).

18. Let BAC be an acute $\angle$, and let BD, CD be perp. to AB and AC respectively. Produce CD to E. Join AD. The $\angle$s of the two $\triangle$s ABD, ACD = 4 rt. $\angle$s (I. 22.) $\therefore$ $\angle$BAC + $\angle$BDC = 2 rt. $\angle$s. But $\angle$EDB + $\angle$BDC = 2 rt. $\angle$s (I. 1.) $\therefore$ $\angle$EDB = $\angle$BAC.

19. $\angle$AMC + $\angle$LMN = 2 rt. $\angle$s (I. 1.) $\angle$BLC + $\angle$MLN = 2 rt. $\angle$s (I. 1.) $\angle$ANB + $\angle$MNL = 2 rt. $\angle$s (I. 1.) $\therefore$ since $\angle$s LMN, MLN, MNL = 2 rt. $\angle$s (I. 22.) $\angle$s AMC, BLC, ANB = 4 rt. $\angle$s.

20. Let M, N be the mid. pts. of AC and BC respectively. From $\triangle$s AMD, CMD, $\angle$DCM = $\angle$DAC (I. 4.). From $\triangle$s BME, CNE, $\angle$NCE = $\angle$EBC (I. 4.) $\therefore$ $\angle$ACB = $\angle$DCE + $\angle$A + $\angle$B $\therefore$ 2$\angle$ACB = $\angle$DCE + $\angle$A + $\angle$B + $\angle$ACB = $\angle$DCE + 2 rt. $\angle$s (I. 22.) $\therefore$ $\angle$DCE = twice the excess, etc.

21. From $\triangle$s DAE, CAE, $\angle$DEA = $\angle$CEA = $\frac{1}{2}\angle$DEC = 30° (I. 5. and 22.). Let AB, CD meet at O. From $\triangle$s DOA, COA, $\angle$DOA = $\angle$COA = a rt. $\angle$ $\therefore$ $\angle$ADO = 30° for $\angle$DAO = 60°. But $\angle$EDO = 60° $\therefore$ $\angle$EDA = 30° $\therefore$ $\angle$DAE = 180° − $\angle$DAE − $\angle$ADE = 120° $\therefore$ $\angle$DAE + $\angle$DAO = 180° $\therefore$ EAB is a str. line (I. 2.).

22. Let ED produced meet CB produced at O. $\angle$ODB = supplement of $\angle$s EDC and BDC = supplement of $\angle$s CED and DBC (I. 5.) = supplement of $\angle$s CED and ACB (I. 5.) = $\angle$DOB. But $\angle$DBC = $\angle$ODB + $\angle$BOD (I. 22.) = 2$\angle$DOB. $\therefore$ $\angle$DOB = $\angle$CBF if BF bisects $\angle$ABC $\therefore$ DE is $\parallel$ to BF (I. 19.).

23. Let the quad[l] ABCD have its sides AB, DC produced to E and F. The four angles of the quad[l] = 4 rt. $\angle$s (Exercises

XIII. 12). But $\angle$EBC + $\angle$ABC + $\angle$BCF + $\angle$BCD = 4 rt. $\angle$s (I. 1.) $\therefore$ $\angle$EBC + $\angle$BCF = $\angle$BAD + $\angle$ADC.

24. Let $\angle$ADE = $\angle$DEA = a°, so that $\angle$DAE = $6a^\circ$, $8a = 180$ $\therefore a = 22\frac{1}{2}$. But $\angle$DAC = $\angle$ACB − $\angle$ADC (I. 22.) = $45^\circ - 22\frac{1}{2}^\circ = 22\frac{1}{2}^\circ$ = $\angle$CDA $\therefore$ CD = CA (I. 6.) $\therefore$ we must produce BC both ways, so that DC = BE = CA or AB.

25. Let the bisectors meet at E. $\angle$BEC = $\angle$ECD − $\angle$EBC (I. 22.) = $\frac{1}{2}$ [$\angle$ACD − $\angle$ABC] = $\frac{1}{2}\angle$BAC (I. 22.).

26. Let the quad[l] ABCD have the side AB held fast. Join AC. From $\triangle$s DAC, CBA, $\angle$DCA = $\angle$CAB (I. 7.) $\therefore$ DC is $\parallel$ to AB in all positions, *i.e.* all positions of DC are parallel (I. 21.).

27. $\angle$A + $\angle$BDC = 2 rt. $\angle$s. = $\angle$BDC + $\frac{1}{2}\angle$B + $\frac{1}{2}\angle$C (I. 22.) $\therefore$ $\angle$A = $\frac{1}{2}\angle$B + $\frac{1}{2}\angle$C = $\frac{1}{2}$[2 rt. $\angle$s − $\angle$A] (I. 22.) $\therefore$ $\angle$A = $\frac{2}{3}$ of a rt. $\angle$ = $\angle$ of an equilateral $\triangle$.

28. 360° = $\angle$B + $\angle$C + $\angle$BDE + $\angle$DEC (Ex. XIII. 12.) = $\angle$B + 2$\angle$DEC (I. 5.) = $\angle$B + 360° − 2$\angle$DEA (I. 1.) $\therefore$ $\angle$B = 2$\angle$DEA = 2$\angle$A (I. 5.).

29. Ext. $\angle$s at A, C, E + ext. $\angle$s at B, D, F = 4 rt. $\angle$s (I. 22, Cor. 2.). Also $\angle$s B, D, F + ext. $\angle$s at B, D, F = 6 rt. $\angle$s (I. 1.) $\therefore$ $\angle$s B, D, F − ext. $\angle$s at A, C, E = 2 rt. $\angle$s.

30. $\angle$DEB = alt. $\angle$EBC (I. 20.) = $\angle$DBE $\therefore$ DE = DB (I. 6.) = DA $\therefore$ $\angle$DEA = $\angle$DAE. Also $\angle$DEB = $\angle$DBE $\therefore$ $\angle$AEB = $\frac{1}{2}$ sum of $\angle$s of $\triangle$AEB = a rt. $\angle$ (I. 22.).

31. $\angle$DEF = $\angle$BAE + $\angle$ABE (I. 22.) = $\angle$EBC + $\angle$ABE (Hyp.) = $\angle$ABC. Similarly, $\angle$DFE = $\angle$ACB and $\angle$EDF = $\angle$BAC.

32. Let ABCD be a quad[l] such that $\angle$A + $\angle$B = 200°. The four $\angle$s of the quad[l] = 360° $\therefore$ $\angle$D + $\angle$C = 160° $\therefore$ if the bisectors of these $\angle$s meet at O in $\triangle$DOC, $\angle$DOC = $180^\circ - \frac{1}{2}$D $- \frac{1}{2}$C = 100°.

33. From $\triangle$s ABD, ACD, BD = DC (I. 16.) $\therefore$ from $\triangle$s BDF, CDF $\angle$FBD = $\angle$FCD (I. 4.) 2$\angle$ACB = 2 rt. $\angle$s − $\angle$A (I. 22.) = $\frac{3}{2}$ rt. $\angle$s $\therefore$ $\angle$ACB = $\frac{3}{4}$ of a rt. $\angle$. Also $\angle$E = a rt. $\angle$ $\therefore$ $\angle$EBC = $\frac{1}{4}$ rt. $\angle$ (I. 22.) $\angle$EFC = 2$\angle$FBC (I. 22.) = $\frac{1}{2}$ rt. $\angle$. Also $\angle$FEC = a rt. $\angle$ $\therefore$ ECF = $\frac{1}{2}$ rt. $\angle$ (I. 22) $\therefore$ EC = EF (I. 6.).

34. Let AD bisect the $\angle$A, and AE be perp. to the base of $\triangle$ABC. 90° − B = $\angle$BAE = $\angle$DAE + $\frac{1}{2}\angle$BAC $\therefore$ adding $\frac{1}{2}\angle$B + $\frac{1}{2}\angle$C to both sides, $90^\circ + \frac{1}{2}\angle$C − $\frac{1}{2}\angle$B = $\angle$DAE + $\frac{1}{2}\angle$s of $\triangle$ABC = $\angle$DAE + 90° (I. 22.) $\therefore$ $\angle$DAE = $\frac{1}{2}\angle$C − $\frac{1}{2}\angle$B.

35. Let AN be perp. to BC. $\angle$ADN $=\angle$ABD $+\angle$BAD $=\angle$B $+\angle$C (I. 22.) $\angle$AEN $=\angle$ACE $+\angle$CAE $=\angle$C $+\angle$B (I. 22.) $\therefore$ $\angle$ADN $=\angle$AEN $\therefore$ from $\triangle$s ADN, AEN, DN $=$ EN (I. 16.).

36. Let ADE be an equilateral $\triangle$, and let DE meet AC at O. From $\triangle$s ABD, ACD, $\angle$BAD $=\angle$CAD $= 30°$. Also $\angle$ADO $= 60°$ $\therefore$ $\angle$AOD $=$ a rt. $\angle$ (I. 22.) $\angle$EAD $= 60°$. Also $\angle$BAD $= 30°$ $\therefore$ $\angle$BAE is a rt. $\angle$. Also AD is perp. to BC by hypothesis.

EXERCISES XIV.

1. Let O be the centre of the given circle, A any point on the locus. Let OA meet the given circle at B. Then OA $=$ sum of radii of the two circles, and is therefore constant $\therefore$ the locus of A is a circle whese centre is O and whose rad. is the sum of the radii of the two circles.

2. In this case, OA $=$ the diff. of the radii of the two circles, and is constant, and the locus is a circle as in Example 1.

3. This locus will be found to be a str. line thro. the intersection of the given lines.

4. This will be found to be a str. line $\parallel$ to the given line.

5. The pt. is at a constant distance from the centre of the given circle; therefore its locus is a concentric circle.

6. Take O the mid. pt. of AB and let OM, ON be drawn perp. to the lines through A and B respectively. $\angle$MAO $=$ alt. $\angle$NBO (I. 20.) $\therefore$ from $\triangle$s AMO, BNO, OM $=$ ON (I. 16.). Thus we see that the locus of O is a str. line equidistant from the given lines.

7. Let AB, AC be the given lines, draw DE $\parallel$ to AB and at a dist. from it equal to the given constant length. Let AC, DE meet in O. Let OP be that part of the bisector of $\angle$AOE which falls within the $\angle$CAB. Taking any pt. F in OP, and drawing FM perp. to AC, and EFN perp. to DE and AB, FM $=$ FE from $\triangle$s OMF, OEF (I. 16.) $\therefore$ FM $+$ FN $=$ EN $\therefore$ F is a pt. on the locus. Similarly if OQ bisect $\angle$DOA and meet BA produced in Q, QO is part of the locus. If CA is produced to R, similar parts of the locus are found within the $\angle$s QAR, RAP, and the complete locus is found to be the perimeter of a right-angled quad[l].

EXERCISES XV.

1. With centre A and rad. AB describe a circle. With centre B and rad. BA describe another circle cutting the first in C and D. △s ABC, ABD will both be equilateral △s on AB.

2. With centres A and B and rad. equal to 2AB describe circles cutting at C and D. △s ACB, AEB both fulfil the reqd. conditions as in I. 25.

3. Let AB be the given base. With centres A and B and rad. equal to the given line, describe circles cutting at C and D. △s ACB, ADB will both fulfil the reqd. conditions as in I. 25.

4. Let A and B be the given pts. With centres A and B and rad. equal to the given radius describe circles. Let C be one of the pts. at which they meet. Then since CA = CB = the given radius a circle described with centre C, and rad. CA is the reqd. circle. The circles will not cut unless the given rad. is gr. than $\frac{1}{2}$AB.

5. This can be done as in the previous exercise.

6. With any pt. O in a str. line AB as centre and any rad. describe a circle cutting AB at A and B. With centres A and B and any rad. gr. than OA, describe circles cutting in C. Join OC. From △s AOC, BOC, ∠AOC = ∠BOC = a rt. ∠ (I. 7.).

7. Join the given pts. A, B. Bisect AB at O (I. 27.) and draw OD at rt. ∠s to AB to meet the given line at D. From △s AOD, BOD, DA = DB (I. 4.) ∴ D is the reqd. pt. The problem is impossible unless OD meets the given line, *i.e.* when OD is ∥ to the given line, *i.e.* when AB is perp. to the given line. When AB is perp. to the given line and bisected by it, any pt. in the given line satisfies the reqd. condition.

8. Draw AD bisecting ∠BAC (I. 26.) and meeting BC at D. If DM . DN be perp. to AB, AC, then from △s AND, AMD, DM = DN (I. 16.) ∴ D is the reqd. pt.

9. With centre C and rad. 1 in., describe a circle. If OD bisects AB at rt. ∠s, any pt. in it is equidistant from A and B (I. 23.) ∴ the pts. where OD meets the circle are the reqd. points. Impossible when OD does not meet the circle.

10. Draw OD bisecting BC at rt. ∠s. Any pt. in OD is equidistant from B and C (I. 23.) ∴ the pt. where OD meets AB is the reqd. pt.

11. Produce BA to D making AD equal to AC. Bisect BD at O. With centre A and rad. equal to OB or OD describe a circle cutting BC at E. $AE = OB = \frac{1}{2} BD = \frac{1}{2}(AB + AC)$ ∴ D is the reqd. pt.

12. Bisect AD, BC at rt. angles by str. lines meeting at E. E is the reqd. pt. (I. 23.).

EXERCISES XVI.

1. Bisect the given angle, and also bisect the two angles thus formed.

2. Let A be the given pt., BC the given line. Thro. A draw AD ∥ to BC (I. 31.). Make ∠DAB equal to the given angle. AB is the reqd. line, for ∠ABC = alt. ∠DAB (I. 16.) = the given angle.

3. Let AB, AC be the given lines, O the given pt. Bisect ∠BAC by AD; and draw ON perp. to AD, and produce it to meet AB, AC in E and F. From △s ANE, ANF ∠AFN = ∠AEN (I. 16.) ∴ EF is the reqd. line.

4. Thro. A the given pt. draw EAD ∥ to the given line and make ∠DAC equal to the given ∠. Bisect ∠EAC by AF meeting BC at F. If FE is ∥ to CA, ∠EFA = alt. ∠FAC (I. 20.) = ∠EAF = alt. ∠AFC. ∠DAC = alt. ∠ACF (I. 20.) = ext. ∠BFE (I. 20.) = ∠BFA − ∠EFA = ∠BFA − ∠AFC ∴ AF is the reqd. line.

5. Let OA, OB, OC be the given lines, OB falling between OA and OC. In OB take any pt. D. Make BD equal to DO, and from B draw BA ∥ to CO to meet OA at A. Join AD and produce it to meet OC at C. ∠ADB = ∠CDO (I. 3.). ∠ABD = alt. ∠DOC (I. 20.) ∴ from △s ADB, CDO, DA = DC (I. 16.).

6. Let BAC be the given ∠, P and Q the given perpendiculars. At A draw AD perp. to AB and equal to P. At A draw AE perp. to AC and equal to Q. Draw DC ∥ to AB to meet AC at C, and draw EB ∥ to AC to meet AB at B. ABC is the reqd. △. For if CM be perp. to AB and BN perp. to AC, ∠DAC = complement of ∠CAM = ∠ACM ∴ from △s, ADC, CMA ∴ CM = DA = P (I. 16.). Similarly, BN = Q, and ∠BAC is the given ∠.

7. Let ABC be a rt. ∠. On BC describe an equilateral △BDC. ∠DBC = $\frac{2}{3}$ of a rt. ∠ (I. 22.) ∴ ∠ABD = $\frac{1}{3}$ of a rt. ∠ ∴ bisecting ∠DBC by BE, BD and BE divide ∠ABC into three equal parts.

8. Let A, B, C be the given pts. On AB describe an equilateral △DBA. Thro. C draw ECF ‖ to AB to meet DA and DB in E and F. ∠DEF = ext. ∠DAB (I. 20.) = 60° = ∠DBA (I. 5.) = int. ∠DFE (I. 20.) and ∠FDE also = 60° ∴ △DEF is equilateral (I. 6.).

9. If a is an angle at the base, $6a = 180°$ and $a = 30°$ ∴ if we bisect two angles of an equilateral △, we obtain the △ reqd.

10. Draw BE ‖ to CA. Make ∠EBD equal to ∠ABC and let BD meet AC at D. ∠BDA = alt. ∠EBD (I. 20.) = ∠ABC.

11. Describe an equilateral △ABC as in I. 25. ∠ABC = 60° (I. 5. and 22.). Bisect ∠ABC by BD (I. 26.) and ∠DBC = 30°. Bisect ∠DBC and we have ∠s of 15°, *i.e.* one-sixth of a rt ∠.

12. Let ∠BAE be the given ∠, AB and P the given sides, ∠BAE being opp. to P. With centre B and rad. P describe a circle cutting AE in C. BAC is the △ reqd. Since the circle will generally cut AE in two pts. we generally obtain two solutions.

13. Let BAC be the given vertical ∠. Bisect it by AD (I. 26.) and make AD equal to the given perp. Draw EDF perp. to AD meeting AB, AC in E and F. From △s ADE, ADF, AE = AF (I. 16.) and AD = given perp. ∴ △AEF is the △ reqd.

14. Let AB be the given perp^r^. Draw BC perp. to AB making BC equal to half the perimeter. Join AC, and make ∠CAD equal to ∠ACD, AD meeting BD at D. In DB produced make BE equal to DB. Join AE. AD = DC (I. 6.) ∴ AD + DB = half the given perimeter. Also from △s ABE, ABD, AE = AD (I. 4.) ∴ ADE is the △ reqd.

15. Let AB be the given side. Draw AC at rt. ∠s to AB. With centre B, and rad. equal to the hypotenuse, describe a circle cutting AC at D. DAB is the △ reqd.

16. From AC the perimeter cut off AB equal to the hypotenuse. At C make ∠BCD equal to half a rt. ∠; and with centre B and rad. BA describe a circle meeting CD at D. Draw DE perp. to

BC. ∠DEC = a rt. ∠, ∠DCE = $\frac{1}{2}$ a rt. ∠ ∴ ∠EDC = $\frac{1}{2}$ a rt. ∠ (I. 22.) ∴ ED = EC (I. 6.). Also BD = the given hypotenuse ∴ DEB is the △ reqd.

17. Let AB be the given perp^r. Draw BC perp. to BA, and on BC describe an equilateral △DBC. From A draw AF ∥ to DB and AE ∥ DC to meet BC in F and E. ∠AFE = ext. ∠DBC = $\frac{2}{3}$ of a rt. ∠. ∠AEF = int. opp. ∠DCE = $\frac{2}{3}$ of a rt. ∠ ∴ ∠FAE = $\frac{2}{3}$ of a rt. ∠ (I. 22.) ∴ △AFE is equilateral (I. 6.), and is the △ reqd.

18. Bisect ∠ABC by BE meeting AC at E. Draw ED ∥ to CB meeting AB at D. ∠DEB = alt. ∠EBC (I. 20.) = ∠DBE ∴ DB = DE (I. 6.) ∴ D is the pt. reqd.

EXERCISES XVII.

1. Use the method of I. 25. The lengths of the sides in cms. are 8·9, 6·35, 10·2.

2. Use the method of I. 25. The lengths of the sides in inches are 1·57, 1·97, 2·36.

3. With a protractor make ∠BAC equal to 35°. Cut off AB equal to 5 cms. and draw BC perp. to AC. BAC is the △ reqd.

4. If we fold the quadl. ABCD about the diagonal BD, the pt. C will fall upon the pt. A ∴ AC is bisected by the crease, for its parts coincide. Similarly BD will be bisected if we fold about AC. Also if AC, BD meet at O, adj. ∠s AOD, COD coincide when we fold, and are therefore rt. ∠s. Hence, the diagonals of an equilateral quadl. (a rhombus) bisect one another at rt. ∠s.

5. Draw AB equal to 2 inches. At A and B, with a protractor, make ∠s CAB, CBA each equal to 40°. ABC is the △ reqd.

6. Each angle of the pentagon = $\frac{6}{5}$ of a rt. ∠ (p. 57) = 108°. Draw AB = 3 cms. and make, with a protractor, ∠s DAB, CBA each equal to 108°. Cut off AD = BC = 3 cms. With centres D and C and radii 3 cms. describe arcs cutting at E. ABCED is the fig. reqd.

7. Draw AB, 4 cms. long. With centres A and B and radii AB, describe circles cutting at O. With centre O and same

rad. describe a circle ABCDEF. Let this circle cut the first two circles at F, etc. With centres F and C and same rad. describe circles cutting circle ABEF at D and E. ABCDEF will be a regular hexagon.

8. Draw AB 2 in. long. At A and B with a protractor make $\angle$CAB $= 30°$ and $\angle$CBA $= 50°$. ABC is the $\triangle$ reqd. The sides are 4 cms. and 2·6 cms. long.

9. Construct an equilateral $\triangle$ and bisect one angle. This will give two $\triangle$s each satisfying the given conditions.

10. Draw a str. line AB, and fold the paper so that the pt. A falls on B. Let COD be the crease, meeting AB at O. When we fold, $\angle$COA coincides with $\angle$COB, and they are adj. angles $\therefore$ they are rt. $\angle$s $\therefore$ the crease CD is perp. to AB.

11. Take any str. line AB. With centres A and B and rad. equal to AB describe circles meeting in C. ABC is an equilateral $\triangle$ $\therefore$ $\angle$CAB $= 60°$ (I. 22.) $\therefore$ bisecting $\angle$CAB we obtain angles of 30°.

12. Draw any str. line AB, and with a protractor make $\angle$BAC equal to 60° and $\angle$ABC equal to a rt. $\angle$. $\angle$BCA $= 30°$ (I. 22.) $\therefore$ ABC is the $\triangle$ reqd. Produce AB to D, making BD $=$ BA. Join CD. From $\triangle$s CBA, CBD, $\angle$CDB $= \angle$CAB $= 60°$, and $\angle$DCB $= \angle$ACB $= 30°$ (I. 4.) $\therefore$ $\triangle$ACD is equilateral (I. 6.) $\therefore$ AC $=$ AD $= 2$ AB.

13. At the mid. pt. of a line 1·4 in. long draw a perp^{r}. 2·4 in. long. Joining the ends of the first line to the end of the perp^{r}. we have the $\triangle$ reqd. (I. 4.). The sides are 2·5 in. long.

14. Make $\angle$ABC $= 33°$ with a protractor. Cut off BA $= 2·5$ in. and BC $= 3·4$ in. ABC is the $\triangle$ reqd. BC $= 1·91$ in.

15. Draw AB, BC at rt. $\angle$s to one another. Cut off BA $= 3·7$ cms., and BC $= 3·2$ cms. ABC is the $\triangle$ reqd. AC $= 4·9$ cms.

16. With centre C and rad. 1 in. describe a circle, meeting AB at E and F. CE and CF both give solutions. A circle generally meets a str. line at two points, and we therefore generally obtain two solutions. If the perp. from C upon AB is gr. than 1 in. a solution is impossible, for in that case the circle does not meet AB.

17. Use the method of I. 25.

18. Let BC represent the tower, A the pt. in the horizontal plane such that $\angle$CAB $= 45°$ and BA $= 50$ ft. $\angle$CBA $= 90°$ $\therefore$ $\angle$ACB $= 45°$ $\therefore$ CB $=$ BA $= 50$ ft.

19. $\angle$ reqd. $= 97°$.

20. Make $\triangle$ABC as in I. 25. Bisect AC at D. (This can be done by making AD $= 2\frac{1}{2}$ cms.) DB $= 2·5$ cms.

21. Draw AB 3 in. long; at A with a protractor make $\angle$BAC $= 40°$, and at B make $\angle$ABC $= 60°$. CA $= 2·64$ in. and CB $=$ 1·96 in.

22. Draw the $\triangle$ as in Example 14 above. The third side $=$ 1·61 in.

23. 4 in. $= 10·16$ cms. $\therefore$ 1 in. $= 2·54$ cms. Any error made in measuring the 4 in. line is divided by 4, in finding 1 in. in cms.

24. Draw AB $= 3$ in., AC perp. to AB, and (with a protractor) $\angle$ABC $= 30°$. AC represents the tower. AC $= 173·2$ ft.

25. Draw BA 3 cms. long to represent 30 feet, and produce it to D. With a protractor, make $\angle$DBC $= 45°$, and $\angle$DAC $= 60°$. Draw CD perp. to BA. CD represents the tower. CD $= 7·1$ cms. $\therefore$ the tower is 71 ft. high.

26. Make AB 3 in. Make $\angle$s BAC, ABC each 40°. Then $\angle$ACB $= 100°$; and by measurement AC $= 1·96$ in. approx. which represents 196 feet.

27. Draw BC 1 in. long to represent the height of the mound. Draw BA perp. to BC, and make BA $= 1$ in. $\angle$CAB $=$ $\angle$ACB $= 45°$ (I. 5. and 22.). With a protractor, make $\angle$BAD equal to 60°, and let BC produced meet BD at D. CD is the flagstaff. AD $= 2$ inches $\therefore$ the dist. of A from the top of the flagstaff $= 40$ feet.

28. With the protractor draw an $\angle$ of 45°. Bisect it and we obtain angles of $22\frac{1}{2}°$.

⁂ **The additional Exercises, 29-39, will be found on pages 173-174.**

EXERCISES XVIII.

1. It is reqd. to draw a perp. to BA at the pt. A. With any centre C and rad. CA describe a circle cutting AB again at B. Join BC and produce it to meet the circle again at D. Join AD. $\angle$CAB $=$ $\angle$CBA (I. 5.) and $\angle$CAD $=$ $\angle$CDA (I. 5.) $\therefore$

$\angle$BAD = $\angle$CBA + $\angle$CBA $\therefore$ $\angle$BAD = $\frac{1}{2}$ sum of the $\angle$s of $\triangle$ ABD = a rt. $\angle$ $\therefore$ AD is the reqd. line.

2. Let AB be the given pts., CD the given line. Draw AF perp. to CD and produce it to G, making FG = AF. Join BG cutting CD at E. Join AE. From $\triangle$s AFE, GFE, $\angle$AEF = $\angle$GEF (I. 4.) = $\angle$BED (I. 3.) $\therefore$ AE, EB make equal angles with CD.

3. (1) Let O be the given pt., PQ the given line, and let OA be perp. to PQ. Draw any other line OB to meet PQ in B. $\angle$BAO = $\angle$QAO > $\angle$OBA (I 8.) $\therefore$ OB > OA (I. 11.). Similarly any other line from O to PQ > OA. (2) Draw OC further from OA than OB to meet PQ in C. $\angle$OBC > $\angle$OAB and is $\therefore$ an obtuse $\angle$ $\therefore$ $\angle$OBC > $\angle$OBA > $\angle$OCB (I. 8.) $\therefore$ OC > OB (I. 11.). (3) On the side of OA remote from OB, let $\angle$AOD = $\angle$AOB. From $\triangle$s OAB, OAD, OD = OB (I. 16.). If possible let OC = OB = OD, OC and OB being on the same side of OA, and OC further from OA than OB. $\angle$OCB = $\angle$OBC (I. 5.) which > $\angle$OAB a rt. $\angle$ (I. 8.) $\therefore$ the two $\angle$s OCB, OBC of $\triangle$ OBC are together gr. than two rt. $\angle$s, which is impossible (I. 9.).

4. Let AD be the line joining A to the mid. pt. D of the side BC of the $\triangle$ ABC. Produce AD to E, making DE equal to AD. $\angle$CDE = $\angle$BDA (I. 3.) $\therefore$ from $\triangle$s CDE, BDA, CE = BA (I. 4.) $\therefore$ BA + AC = EC + CA > AE (I. 12.) $\therefore$ BA + AC > 2AD.

5. Let AB be the given base. Bisect it at C and draw CD at rt. $\angle$s to AB and equal to the given sum. Join AD and make $\angle$DAE equal to $\angle$CDA, AE meeting CD at E. Join EB. From $\triangle$s ACE, BCE, AE = BE (I. 4.). Also AE = DE (I. 6.) $\therefore$ AE + EC = CD = given sum $\therefore$ AEB is the $\triangle$ reqd.

6. Let AB be the given base, BAC the given $\angle$, and AC the given sum of the two sides. Join BC, and make $\angle$CBD equal to $\angle$ACB, BD meeting AC at D. DB = DC (I. 6.) $\therefore$ AD + DB = AC = given sum of sides $\therefore$ ADB is the reqd. $\triangle$.

7. Let DE be the given perimeter. At D and E make $\angle$s EDA, DEA equal to half the given angles at the base. At A, where DA and EA meet, make $\angle$DAB equal to $\angle$ADB, and $\angle$EAC equal to $\angle$AEC, AB and AC meeting DE at B and C. AB = DB and CA = CE (I. 6.) $\therefore$ AB + BC + CA = DE = given perimeter. Also $\angle$ABC = 2$\angle$ADB = given base angle (I. 22.). Similarly $\angle$BCA = the other given base angle $\therefore$ ABC is the $\triangle$ reqd.

8. Let A, B be the given pts., CD the given line. Draw AE perp. to CD and produce AE to F, making EF equal to EA. Join BF cutting CD at G. Join AG. Take any other pt. H in CD, and join FH, BH. From $\triangle$s AEG, FEG, AG = FG. (I. 4.) $\therefore$ FH + HB > FB (I. 12.) > AG + GB. Thus we see that AG + GB is a minimum.

9. Let D be the mid. pt. of the hypotenuse AB of the $\triangle$ ABC. Join DC. If DC > DB or DA, $\angle$DBC > $\angle$DCB and $\angle$DAC > $\angle$DCA (I. 10.) $\therefore$ $\angle$ABC + $\angle$BAC > $\angle$BCA $\therefore$ $\angle$BCA < a rt. $\angle$ (I. 22.), which is contrary to the hypothesis. Similarly, if DC < DB it may be proved from I. 22. that $\angle$BCA > a rt. $\angle$, which is contrary to the hypothesis $\therefore$ DC must be equal to DB or DA.

EXERCISES XIX.

1. With the fig. of II. 1 let AD, BC meet in O. $\angle$AOB = $\angle$DOC (I. 3.). $\angle$ABO = alt. $\angle$OCD (I. 20.) $\therefore$ from $\triangle$s AOB, COD, AO = OD and BO = OC (I. 16.).

2. Let ABCD be a quadl. having its opp. sides equal. From $\triangle$s ABD, CDB, $\angle$ABD = $\angle$CDB and $\angle$ADB = $\angle$CBD (I. 7.) $\therefore$ AB is $\parallel$ to CD, and AD to BC (I. 18.) $\therefore$ ABCD is by def. a parm.

3. Let ABCD be the quadl. Its four $\angle$s are equal to four rt. $\angle$s (I. 22.) $\therefore$ $\angle$DAB + $\angle$ADC = 2 rt. $\angle$s $\therefore$ AB is $\parallel$ to CD (I. 19.). Similarly AD is $\parallel$ to BC $\therefore$ the fig. is a parm.

4. Let AB be the given line. Draw AC, BD $\parallel$ to one another by I. 31. Also AD and BC $\parallel$ to one another. ADBC is a parm. $\therefore$ its diagonals AB, CD bisect one another (II. 2., Cor. 3.).

5. Let ABCD be the parm. and let its diagonals cut at O. $\angle$s BAD, ADC = 2 rt. $\angle$s (I. 20.) $\therefore$ $\angle$DAO + $\angle$ADO = a rt. $\angle$ $\therefore$ AOD = a rt. $\angle$ (I. 22.) $\therefore$ from $\triangle$s AOD, AOB, AD = AB (I. 16.) = CD = BC $\therefore$ ABCD is a rhombus.

6. Let ABCD be a rhombus. Draw its diagonals. From $\triangle$s ADB, CBD, $\angle$ABD = $\angle$BDC (I. 7.) $\therefore$ AB is $\parallel$ to CD (I. 18.). Similarly AD is $\parallel$ to BC.

7. Let ABCD be the parm. From $\triangle$s DAB, CBA, $\angle$DAB = $\angle$CBA (I. 7.). But $\angle$DAB + $\angle$CBA = 2 rt. $\angle$s (I. 20.) $\therefore$ $\angle$DAB is a rt. $\angle$ $\therefore$ ABCD is a rectangle.

8. Let the diagonals of ABCD bisect one another at O. $\angle$AOD = $\angle$BOC (I. 3.) $\therefore$ from $\triangle$s AOD, COB, $\angle$ABD = $\angle$BDC (I. 4.) $\therefore$ AB is $\parallel$ to CD (I. 18.). Similarly AD is $\parallel$ to BC.

9. Let ABCD be the quadl. formed by the rails. Draw AE perp. to BC and AF to CD. AE = AF. In $\triangle$s AFD, AEB, AE = AF, $\angle$AFD = a rt. $\angle$ = $\angle$AEB, and $\angle$ADF = $\angle$ABE (II. 2.) $\therefore$ AD = AB (I. 16.). Also AD = BC and AB = CD $\therefore$ ABCD is a rhombus.

10. Let the diagonals of the rhombus ABCD intersect at O. From $\triangle$s DAC, BAC, $\angle$DAC = $\angle$BAC (I. 7.) $\therefore$ from $\triangle$s DAO, BAO, DO = OB, and $\angle$AOD = $\angle$AOB = a rt. $\angle$ (I. 4.). Similarly AO = CO.

11. From BC cut off BD equal to the given line, and thro. D draw DE $\parallel$ to BA to meet CA at E. Draw EF $\parallel$ to BC. By constr. FBDE is a parm. $\therefore$ FE is $\parallel$ and equal to BD, which is equal to the given line.

12. CD and EF are each $\parallel$ and equal to AB $\therefore$ they are themselves equal and $\parallel$ (I. 21.) $\therefore$ FECD is a parm. (II. 1.).

13. Let ABCD be the parm. such that AB = 2AD. Bisect AB at E. Join ED, EC. Also join E to the mid. pt. F of CD. AEFD is a parm. (II. 1.) $\therefore$ EF = AD = DF = CF (II. 2.) $\therefore$ $\angle$FDE = $\angle$FED and $\angle$FCE = $\angle$FEC $\therefore$ $\angle$DEC = $\angle$FDE + $\angle$FCE $\therefore$ DEC is a rt. $\angle$ (I. 22.).

14. Let ACBD be the quadl. formed by joining the ends of the diameters AOB, COD. OD = OA = OC $\therefore$ $\angle$OAD = $\angle$ODA, and $\angle$OAC = $\angle$OCA (I. 5.) $\therefore$ $\angle$DAC = $\angle$ADC + $\angle$ACD $\therefore$ $\angle$DAC = a rt. $\angle$ (I. 22.). Also from $\triangle$s AOD, BOC, $\angle$DAO = $\angle$CBO and AD = BC (I. 3. 4.) $\therefore$ AD is equal and $\parallel$ to BC $\therefore$ ADBC is a rectangle.

15. Join EG. $\angle$HAE = $\angle$FCG (II. 2.) $\therefore$ from $\triangle$s HAE, FCG, HE = FG (I. 4.). Similarly HG = EF $\therefore$ from $\triangle$s EHG, GFE, $\angle$HEG = $\angle$EGF $\therefore$ HE is $\parallel$ to FG (I. 18.), and it is also equal to it $\therefore$ EFGH is a parm (II. 1.).

16. Let AB be $\parallel$ to CD and AD equal, but not $\parallel$, to BC in the quadl. ABCD. Draw AE $\parallel$ to BC. AEBC is a parm. $\therefore$ AE = BC (II. 2.) = AD $\therefore$ $\angle$ADE = $\angle$AED (I. 5.) = int. opp. $\angle$BCD (I. 20.). Also $\angle$DAB + $\angle$ADC = 2 rt. $\angle$s = $\angle$CBA + BCD (I. 20.) $\therefore$ $\angle$DAB = $\angle$CBA.

17. Let ABCD be the given parm., and let its diagonals cut at O. They bisect one another (II. 2. Cor. 3.). Let E in AB be the given vertex. Let EO produced meet CD in F, and draw

GOH perp. to EF to meet AD in G and BC in H. ∠AOE = ∠COF (I. 3.). ∠AEO = ∠OFC (I. 20.) ∴ from △s AOE, COF, OE = OF (I. 16.). Similarly from △s AOG, COH, OG = OH. Hence from △s GOE, GOF, GE = GF (I. 4.). Similarly HF = HE = EG ∴ EHFG is a rhombus, and is described as reqd.

18. Let ABCD be the given parm., and let its diagonals meet at O. Let P be the given pt. thro. which a diagonal of the reqd. rhombus passes. Join OP, and let it when produced meet AB at E and CD at F. Drawing GOH perp. to EF to meet AD at G and BC at H, it may be proved, as in Example 17 above, that EHFG is the rhombus reqd.

19. From P a pt. in the base BC of the isos. △ABC let PM, PN be drawn perp. to AB and AC respectively. Draw BK ∥ to AC, and let NP produced meet it at K. ∠BKP = alt. ∠KNC (I. 20.) = a rt. ∠ ∴ KN is the perp. dist. between the ∥ lines BK, AC, and is therefore constant in length. Also ∠PBK = alt. ∠PCN (I. 20.) = ∠ABP (I. 5.) ∴ from △s PMB, PKB, PM = PK (I. 16.) ∴ PM + PN = PK + PN = constant.

20. Join AD, and produce it to E, making DE = DA. Draw EC ∥ to AB. Also join CD, and let it when produced meet AB at B. ∠EDC = ∠ADB (I. 3.). ∠ECD = alt. ∠ABD (I. 20.) ∴ from △s ABD, ECD, BD = DC (I. 16.).

21. Let EF meet AD at O. From △s AOF, AOE, AF = AE and OF = OE (I. 16.). Also ∠EDA = alt. ∠DAF (I. 20.) = ∠DAE ∴ from △s AOE, DOE, AO = DO (I. 16.).

22. Let the diagonals of the parm. ABCD cut at O. AO = OC and BO = OD (II. 2. Cor. 3.). ∠AOD = ∠BOC ∴ △AOD = △BOC (I. 4.). ∠AOE = ∠COF (I. 3.). ∠EAO = ∠FCO (I. 20.) ∴ △AOE = △COF (I. 16.). Similarly △DOF = △EOB ∴ fig. ADFE = fig. EBCF.

23. Draw GCH ∥ to BDF to meet AB at G and EF at H. ∠ACG = ∠HCE (I. 3.). ∠AGC = alt. ∠EHC (I. 20.) ∴ from △s ACG, ECH, CG = CH (I. 16.). Also GCDB, CHFD are parms. ∴ CG = BD and CH = DF (II. 2.) ∴ BD = DF.

24. Let the bisectors AE, DE of two angles of the quadl. ABCD meet at E. AED is a rt. ∠ by hyp. ∴ ∠DAE + ∠ADE = a rt. ∠ (I. 22.) ∴ ∠BAD + ∠ADC = 2 rt. ∠s ∴ AB is ∥ to CD (I. 19.). Similarly AD is ∥ to BC ∴ ABCD is a parm.

25. Let ABCD be a quadl. having AB equal to CD and obtuse ∠DAB equal to obtuse ∠BCD. Join BD. Then in △s DAB, BCD, AB = CD, DB is common, and ∠DAB = ∠DCB ∴ ∠s ADB, DBC are either equal or supplementary (Prop. p. 44). But each of these ∠s is < a rt. ∠ (I. 22.) ∴ they cannot be supplementary ∴ ∠ADB = ∠CBD ∴ ∠ABD = ∠CDB (I. 22). ∴ AB is equal and ∥ to CD (I. 20.) ∴ ABCD is a parm.

EXERCISES XX.

1. Let D, E, F be the mid. pts. of the sides BC, CA, AB of △ABC. Join CF, BE. AE = EC ∴ △EBC = △AEB = $\frac{1}{2}$△ABC (II. 6.). Similarly △FBC = $\frac{1}{2}$△ABC ∴ △EBC = △FBC ∴ EF is ∥ to BC (II. 7.). Similarly DF is ∥ to AC, and DE to AB. Hence EFBD is a parm. ∴ EF = BD = $\frac{1}{2}$BC. Similarly DE = $\frac{1}{2}$AB and DF = $\frac{1}{2}$AC.

2. Let E, F, G, H be the mid. pts. of the sides AB, BC, CD, DA of the quadl. ABCD. EH is ∥ to BD and equal to $\frac{1}{2}$BD by the above example. Similarly FG is ∥ to BD and equal to $\frac{1}{2}$BD ∴ EH and FG are equal and ∥ (I. 21.) ∴ EFGH is a parm. (II. 1.). Also its diagonals bisect one another (II. 2. Cor. 3.), and this proves the third part of the exercise.

EXERCISES XXI.

1. Let ABCD, ABEF be equal parms. on the same base AB and on the same side of it. If DCFE is not a str. line, produce DC to meet AF and BE in G and H. Parm. ABHG = parm. ABCD (II. 3.) = parm. ABEF Hyp., the part equal to the whole, which is impossible ∴ CDFE must be a str. line, *i.e.* the parms. ABCD, ABEF are between the same parallels.

2. Let ABCD be the given rhombus. Bisect BC at E, and AD at F, and join EF, thus dividing the rhombus into two parallelograms (II. 1.). Produce FE to H making FH equal to one-half the given perimeter. Bisect EH at K. With centre E and rad. EK describe a circle meeting AB at P. Draw FQ ∥ to EP to meet AB at Q. Produce PE, QF to meet CD at M and N. FQ + PQ + PE = FH and parm. QE = parm. AE. Also from △s PEB, MCE, EM = EP (I. 3. 20. 16.) ∴ FN + NM + ME = FH ∴ parm. PQNM has the reqd. perimeter and is equal to the rhombus in area.

3. Bisect the sides AB, CD of the parm. ABCD at E and G. Join EG. EG is || to BC and AD (II. 1.) ∴ AG and EC are parms. Also we see by II. 4. that these parms are equal.

4. Let ABCD, EFGH be two equal parms. between the same ||s BCFG, ADEH. If FG is not equal to BC, make FK equal to BC and draw KM || to EF or GH to meet AH at M. Parm. EFGH = parm. ABCD = parm. EFKM (II. 4.), the part equal to the whole, which is impossible ∴ the bases must be equal.

5. Produce HB to meet AC at M and KL at N. Also produce KA to meet EH at P, and LC to meet FG at Q. Parm. AN = parm. BP (II. 4.) = parm. ABDE (II. 3.). Similarly parm. CN = parm. BQ = parm. BCGF ∴ Parm. AKLC = parm. ABDE + parm. BFGC.

6. Let ABC, DEF be equal △s of equal altitudes, AB, DE being their bases. Place the △ABC so that A falls on D and AB along DE, the △s being on the same side of DE. Let KDH be the new position of △ABC. Since the altitudes are equal KF is || to DH ∴ △DEF = △ABC = △KDH = △FDH (II. 5.), the part equal to the whole unless the point H falls on the pt. E ∴ the pt. H must fall on the pt. E ∴ AB = DE.

7. ∠ADO = ∠CBO (I. 20.), ∠AOD = ∠COB (I. 3.) ∴ △AOD = △COB (I. 16.). Also △AOD = △COD (II. 6.) ∴ △COB = △COD.

8. Let AC, BD cut at O. Draw DM, BM perp. to AC. DO = BO (II. 2. Cor. 3.). ∠DON = ∠BOM ∴ from △s DON, BOM, DN = BM ∴ △s BPQ, DPQ are on the same base and have equal altitudes ∴ they are equal in area.

9. Let AB be the given base, CD || to AB, ACB an isos. △. Produce AC to E, making CE equal to CA. Join DE, DB. ∠ECD = int. opp. ∠CAB (I. 20.) = ∠CBA (I. 5.) = alt. ∠BCD (I. 20.) ∴ from △s DCE, DCB, DE = DB ∴ AD + DB = AD + DE > AE (I. 12.), *i.e.* > AC + CB.

10. Let E, F, G, H be the mid. pts. of the sides AB, BC, CD, DA of the parm. ABCD. EB = CG and is || to it ∴ EBCG is a parm. ∴ △EGF = △EBG (II. 5.) = $\frac{1}{2}$ parm. EBCG (II. 2.). Similarly △EHG = $\frac{1}{2}$ parm. EADG ∴ EFGH = $\frac{1}{2}$ parm. ABCD.

11. AD = DB ∴ △ADC = △DBC = $\frac{1}{2}$ △ABC (II. 6.). Also AE = EC ∴ △BEC = △BEA = $\frac{1}{2}$△ABC (II. 6.) ∴ △BEC = △ADC. Take away the common △EFC. Then △BFC = quadl. ADFE.

12. Let AC, BD meet at O. Draw BM, DN perp. to AC. △s ABC, ADC on the same base AC are equal ∴ their altitudes AN, BM are equal ∴ from △s AON, BOM, DO = BO (I. 3. and 16.).

13. Let AC be greater than BC. From AC cut off AD equal to BC. Join PD. △PAD = △PCB (II. 6.) ∴ △PDC is constant in area and on a fixed base DC ∴ its altitude is constant, *i.e.* P lies on one of two str. lines ∥ to AB and at a constant distance from it.

14. △ABC is fixed and the area of ABCD is constant ∴ the area of △ADC is constant, and the △ is on a fixed base AC ∴ D lies on a str. line ∥ to AC and at a constant distance from it.

15. Let the diagonals of the quadl. ABCD meet at O. △AOD = △COB. Add the △COD ∴ △ADC = △BDC ∴ AB is ∥ to DC (II. 7.). Similarly AD is ∥ to BC ∴ ABCD is a parm.

16. AEDF is a parm. ∴ △AED = △AFD (II. 2.). Also △ABD = △ACD (II. 6.) ∴ △EBD = △FDC, and they are on equal bases ∴ EF is ∥ to BC (II. 8).

17. Let ABC, DEF be two △s on equal bases BC, EF, but such that the altitude AM of △ABC = twice DN, the alt. of △DEF. Bisect AM at H, and join BH, CH. △AHB = △BMH and △AHC = △CMH (II. 6.) ∴ △ABC = 2△BHC = 2△DEF (II. 6.).

18. Join DE, and let AE meet BD at N. △DEB = △DCB (II. 5.) = △ADB (II. 2.) ∴ their altitudes EN, AN are equal ∴ from △s ENB, ANB, EB = AB (I. 4.) = CD.

19. Let F be the other pt. of trisection of AC and G the mid. pt. of BD. E, F, G are the mid. pts. of the sides of △ABD ∴ EF is ∥ to BD, EG to AD, and FG to AB (Exercises xx. 1.) ∴ EG is equal and ∥ to DC ∴ from △s DOC, GOE, DO = OG (I. 16.) = $\frac{1}{2}$GD = $\frac{1}{4}$BD.

20. Take G the mid. pt. of BC. EF is ∥ to BC, EG to AB, FG to AC (Exercises xx. 1.) ∴ AFGE, BGEF, CGFE are parms. ∴ △AEF = △FEG = △GEC = △BFG (II. 2.) = $\frac{1}{4}$△ABC.

21. Draw FG ∥ to CD or AB to meet AD at G. △CFE + △DFC = △DEC = $\frac{1}{2}$ parm. ABCD (II. 9.) = $\frac{1}{2}$ parm. DGFC + $\frac{1}{2}$ parm. GABF = △DCF + △ABF (II. 9.) ∴ △CFE = △ABF ∴ adding △BFE to each △CBE = AFE.

22. From △s AEB, FEC, AB = CF (I. 3. 20. and 16.) ∴ DF = 2 . AB (II. 2.) ∴ parm. GDFA = 2 parm. ABCD (II. 4.).

23. Draw FOE || to AB or CD to meet AD at E and BC at F. $\triangle$OAB + $\triangle$COD = $\frac{1}{2}$ parm. AF + $\frac{1}{2}$ parm. EC (II. 9.) = $\frac{1}{2}$ parm. ABCD = $\triangle$ADC (II. 2.) = $\triangle$AOD + $\triangle$AOC + $\triangle$DOC $\therefore$ $\triangle$OAB = $\triangle$AOD + $\triangle$AOC, *i.e.* $\triangle$AOC = the diff. between $\triangle$s OAB and AOD.

24. Let TS, RP meet at O. Draw SN and TM perp. to RP. $\triangle$s PTR, PSR are on same base and equal in area $\therefore$ their altitudes are equal, *i.e.* SN = TM $\therefore$ from $\triangle$s SON, TOM, SO = OT (I. 3. and 16.).

25. Having drawn the $\triangle$ABC as in (I. 25.) bisect BC at D. AD is perp. to BC (I. 7.). AD by measurement = 2·6 in. Thro. A draw AE || to BC, and thro. B draw BE || to DA to meet AE at E. Parm. EBDA = 2$\triangle$BDA (II. 9.) = $\triangle$ABC (II. 6.).

26. Let CD be perp. to AB the base. The area of $\triangle$ABC = $\frac{1}{2}$ the product of AB and CD, and is therefore greatest when CD is greatest. In the rt. $\angle$d. $\triangle$CDB, CB the hypotenuse is the greatest side, and CD increases as we increase the $\angle$CBD $\therefore$ the $\triangle$ is greatest when CD coincides with CB, *i.e.* when $\angle$ABC is a rt. $\angle$.

27. Let ABC be the $\triangle$, O the pt. 1 ft. from BC and AC, x ft. the reqd. dist. of O from AB. $\triangle$AOB + $\triangle$AOC + $\triangle$BOC = $\triangle$ABC $\therefore$ $\frac{1}{2}x \times 5 + \frac{1}{2}4 \times 1 + \frac{1}{2}3 \times 1 = \frac{1}{2} \times 3 \times 4$, whence $x = 1$.

28. Draw EPF || to AB or CD to meet AD at E and BC at F. $\triangle$APB + $\triangle$DPC = $\frac{1}{2}$ parm. AF + $\frac{1}{2}$ parm. EC (II. 9.) = $\frac{1}{2}$ parm. ABCD.

29. $\triangle$ABC = $\triangle$BCE and they have the same altitude $\therefore$ AC = CE. Also, CD is || to BE $\therefore$ $\triangle$BDC = $\triangle$EDC (II. 5.) = $\triangle$ACD for AC = CE (II. 6.) $\therefore$ BD = DA, since $\triangle$s BDC, ACD have the same altitude.

30. Let OL, OM, ON be perp. to the sides BC, CA, AB of the equilateral $\triangle$ABC. $\triangle$ABC = $\triangle$BOC + $\triangle$AOC + $\triangle$AOB = $\frac{1}{2}$OL . BC + $\frac{1}{2}$OM . AC + $\frac{1}{2}$ON . AB = $\frac{1}{2}$BC(OL + OM + ON) $\therefore$ OL + OM + ON is constant.

EXERCISES XXII.

1. $AB^2 = BD^2 + AD^2$ (II. 11.). $AC^2 = CD^2 + AD^2$ (II. 11.) $\therefore$ $AB^2 \backsim AC^2 = BD^2 \backsim CD^2$.

2. Let AD be perp. to the base BC of $\triangle$ABC. From $\triangle$s ABD, ACD, BD = DC = $\frac{1}{2}$BC (I. 5. and 16.). Also $AD^2 + BD^2 = AB^2$ $\therefore$ $AD^2 + \frac{AB^2}{4} = AB^2$ $\therefore$ $AD^2 = \frac{3}{4}AB^2$ $\therefore$ $AD = \frac{\sqrt{3}}{2} . AB$.

3. Let the diagonals of ABCD meet at rt. $\angle$s at O. $AB^2 + CD^2 = (AO^2 + BO^2) + (DO^2 + CO^2)$ (II. 11.) $= (AO^2 + OD^2) + (BO^2 + CO^2) = AD^2 + BC^2$ (II. 11.).

4. $\angle DCA =$ a rt. $\angle = \angle FCB$ $\therefore$ adding $\angle FCD$ to each, $\angle DCB = \angle ACF$ $\therefore$ from $\triangle$s DCB, ACF, $AF = BD$ (I. 4.).

5. Draw AB 3 in. long, and BC, at rt. $\angle$s to it, 2 in. long. $AC^2 = AB^2 + BC^2$ (II. 11.) $= 9 + 4 = 13$ sq. in. $\therefore$ the sq. on AC is the sq. reqd.

6. Draw DL perp. to BC, DM to BF, and DN to CG. From $\triangle$s DCN, DCL, $DN = DL$ (I. 16.). From $\triangle$s DBM, DBL, $DM = DL$ (I. 16.) $\therefore$ $DN = DM$ $\therefore$ from the rt. $\angle$d. $\triangle$s DNA, DMA, $\angle DAN = \angle DAM$ (I. 17.).

7. $CE^2 + BD^2 = (EA^2 + AC^2) + (BA^2 + AD^2)$ (II. 11.) $= (EA^2 + AD^2) + (AC^2 + BA^2) = DE^2 + BC^2$ (II. 11.).

8. Let AE and BK cut at O, and let BK cut AC at F. As in (II. 11.) $\triangle$s ECA, BCK are equal in all respects $\therefore$ $\angle EAC = \angle BKC$. Also $\angle OFA = \angle CFK$ (I. 3.) $\therefore$ $\angle FOA = \angle FCK$ (I. 22.) $=$ a rt. $\angle$.

9. (1) If P, Q, R, S be the pts. of trisection of AB, BC, CD, DA nearest to A, B, C, D respectively. From $\triangle$s SAP, QBP, $PS = PQ$, and $\angle APS = \angle BQP$ (I. 4.) $=$ complement of $\angle BPQ$ (I. 22.) $\therefore$ SPQ is a rt. $\angle$ (I. 1.). In the same way $\angle PQR = \angle QRS = \angle RSP =$ a rt. $\angle$ $\therefore$ PQRS is a parm. (I. 19.). It is also a rect. with two adj. sides equal $\therefore$ it is a sq.

9. (2) Let P, S be the pts. of trisection of AB, AD nearest to A, and Q, R the pts. of trisection of CB, CD nearest to C. $AS = AP$ $\therefore$ $\angle APS = \angle ASP = \frac{1}{2}$ a rt. $\angle$ (I. 22.). Similarly $\angle BPQ = \frac{1}{2}$ a rt. $\angle$ $\therefore$ $\angle SPQ =$ a rt. $\angle$ $\therefore$ PQRS is a rectangle.

10. Draw DH perp. to FB produced $\angle DBH =$ complement of $\angle HBC = \angle CBA$. $\angle DHB =$ a rt. $\angle = \angle BAC$, and $BD = BC$ $\therefore$ $\triangle DBH = \triangle ABC$ and $BH = BA = BF$ (I. 16.) $\therefore$ $\triangle FBD = \triangle HBD$ (II. 6.) $= \triangle ABC$.

11. From $\triangle$s SAP, PBQ, $PS = QR$ (I. 4.). Also $\angle SPA = \angle PQB$ $=$ complement of $\angle BPQ$ $\therefore$ $\angle SPQ$ is a rt. $\angle$. Similarly $\angle$s at Q, R, S are rt. $\angle$s $\therefore$ PQRS is a parm. (I. 19.) rt. $\angle$d. and having two adj. sides equal $\therefore$ it is a sq. (Def.). Moreover $SP^2 = SA^2 + AP^2$ (II. 11.) $= 5 \,.\, AB^2$, for $AP^2 = 4\,AB^2$.

12. Join AO, BO, CO. $AO^2 + BO^2 + CO^2 = (AF^2 + OF^2) + (BD^2 + OD^2) + (CE^2 + OE^2)$ (II. 11.) $= AF^2 + BD^2 + CE^2 + OD^2 + OE^2 +$

OF^2. Also $AO^2 + BO^2 + CO^2 = (AE^2 + OE^2) + (BF^2 + OF^2) + (CD^2 + OD^2) = AE^2 + BF^2 + CD^2 + OD^2 + OE^2 + OF^2$ $\therefore$ $AF^2 + BD^2 + CE^2 = AE^2 + BF^2 + CD^2$.

13. Let DF be drawn perp. to AC produced and DH perp. to AB produced. $\angle CBA$ = complement of $\angle DBH$ (I. 1.) $= \angle BDH$ $\therefore$ from $\triangle$s DBH, BCA, $HB = CA$ (I. 16.). DFAH is a rectangle $\therefore$ $DF = AH = AB + BH = AB + AC$.

14. Let DF be drawn perp. to BE, and $\therefore$ perp. to AD, and $\parallel$ to AB. $\angle ACD = \angle ADC = \frac{1}{2}$ a rt. $\angle$ (I. 5. and 22.) $\therefore$ $\angle FDC$ = alt. $\angle DCA = \frac{1}{2}$ a rt. $\angle$ $\therefore$ $\angle EDF = \frac{1}{2}$ a rt. $\angle$ $\therefore$ $\angle DEF = \frac{1}{2}$ a rt. $\angle$ $\therefore$ $DE^2 = DF^2 + EF^2$ (II. 11.) $= 2 . DF^2$ (I. 6.) $= 2 . AB^2 = 8 . AC^2 = 4 . CD^2$ $\therefore$ $DE = 2CD$.

15. If the diagonals AC, BD of the quadl. meeting at O are not at rt. $\angle$s, let AN, CM be drawn perp. to BD. $AB^2 + CD^2 = (AN^2 + BN^2) + (DM^2 + CM^2)$ (II. 11.). Also $AD^2 + BC^2 = (AN^2 + DN^2) + (CM^2 + BM^2)$ (II. 11.) $\therefore$ $BN^2 + DM^2 = DN^2 + BM^2$ $\therefore$ $BN^2 - BM^2 = DN^2 - DM^2$ ... (1). But if $BN > BM$, $DN < DM$ $\therefore$ (1) is impossible unless the pts. N, M coincide, in which case AC is perp. to BD.

16. $10^2 - 8^2 = (10 - 8)(10 + 8) = 6^2$ $\therefore$ $10^2 = 8^2 + 6^2$ $\therefore$ the lengths form a rt. $\angle$d. $\triangle$ II. 12.

17. $39^2 - 36^2 = (39 - 36)(39 + 36) = 3 \times 75 = 15^2$ $\therefore$ $39^2 = 36^2 + 15^2$ $\therefore$ the lines form a rt. $\angle$d. $\triangle$ (II. 12.).

18. $100^2 - 96^2 = (100 - 96)(100 + 96) = 4 \times 196 = 4^2 \times 49 = 28^2$ $\therefore$ $100^2 = 96^2 + 28^2$ $\therefore$ the lines form a rt. $\angle$d. $\triangle$ (II. 12.).

19. Let ABC be a $\triangle$ such that $AC^2 > AB^2 + BC^2$. Let BD be drawn perp. to AB and equal to BC. $AC^2 > AB^2 + BC^2 > AB^2 + BD^2 > AD^2$ (II. 11.) $\therefore$ $AC > AD$ $\therefore$ from $\triangle$s ABC, ABD $\angle ABC > \angle ABD$ (I. 15.) *i.e.* $\angle ABC >$ a rt. $\angle$.

20. With the same construction as in the preceding example, $AC < AD$ $\therefore$ $\angle ABC < ABD$ (I. 15.) $<$ a rt. $\angle$.

21. Let BE, CF, BK be diagonals of the sqs. on the sides AB, AC, BC of $\triangle ABC$, rt. $\angle$d. at C. $CF^2 = 2 . AC^2$, $BK^2 = 2BC^2$ and $BE^2 = 2BA^2$ (II. 11.) $\therefore$ $CF^2 + BK^2 = 2 . AC^2 + 2BC^2 = 2AB^2$ (II. 11.) $= BE^2$ $\therefore$ the $\triangle$ is rt. $\angle$d. (II. 12.).

EXERCISES XXIII.

1. Let ABC be the given △, D the given pt. in AB. Bisect CA at E, and draw BF ∥ to DE to meet AC at F. Join DF. △ADF = △ADE + △DEF = △ADE + △DBE (II. 5.) = △ABE = $\frac{1}{2}$△ABC for AE = EC. This construction fails if AD < BD. In such a case, BC must be bisected instead of AC.

2. Draw BE ∥ to DA to meet CA at E. Join DE. △CDE = △ADC + △EDA = △ADC + △ADB (II. 5.) = △ABC.

3. Let AB be the given str. line. Draw another str. line ACDE making AC = CD = DE. Join BE and draw DG and CF ∥ to BE to meet AB at G and F. Draw FH, GK ∥ to AE to meet GD, BE. FD and GE are parms. ∴ FH = CD = AC, and GK = DE = CD (II. 2.) ∴ from △s ACF, FHG, AF = FG (I. 20. and 16.). Similarly BG = FG = AF.

4. Use the method of the above exercise.

5. Let the two medians BD, CE of △ABC meet at G. Produce AG to H, making GH = GA. E, G are the mid. pts. of AB and AH ∴ EG is ∥ to BH (Exercises xx. 1). Similarly GD is ∥ to CH ∴ BGCH is a parm. Also its diagonals bisect one another ∴ AG passes through the mid. pt. of BC.

6. Let AO, BO bisectors of angles of △ABC meet at O. Draw OL perp. to BC, OM to AC, ON to AB. From △s AON, AOM, ON = OM (I. 16.). Similarly from △s BON, BOL, ON = OL (I. 16.) ∴ OM = OL ∴ from △s COM, COL, ∠OCM = ∠OCL (I. 17.).

7. Let D, E, F be the mid. pts. of the sides BC, CA, AB of △ABC. Let the perps. at D and E meet at O. Join OF. [The perps. DO, EO must meet, for if they were parallel CB and CA would also be parallel.] From △s BDO, CDO, BO = CO (I. 4.). From △s AOE, COE, AO = CO (I. 4.) ∴ AO = BO ∴ from △s AFO, BFO, ∠AFO = ∠BFO = a rt. ∠ (I. 7.).

8. (1) Let AD be less than BE. Draw HCK ∥ to DFE to meet DA produced at H and EB at K. ∠BCK = ∠ACH (I. 3.). ∠BKC = ∠CHA (I. 20.) ∴ from △s BCK, ACH, BK = AH ∴ AD + BE = (HD − HA) + (KE + BK) = HD + KE = 2CF (II. 2.). (2) Let DFE cut AB between A and C. Draw HCK ∥ to DFE to meet AD produced in H, and BE in K. As in the above BK = AH ∴ BE − AD = BK + KE − (AH − DH) = EK + DH = 2CF (II. 2.).

9. Let AB be the given str. line. At A and B make $\angle$s DAB, ABD each equal to half a rt. $\angle$. Bisect $\angle$ABD by BC meeting AD at C. Draw CE perp. to CA to meet AB at E. E is the reqd. pt. For, $\angle AEC = \frac{1}{2}$ a rt. $\angle$ (I. 22.) $= \angle ECB + \angle EBC$ (I. 22.) $\therefore \angle ECB = \frac{1}{4}$ rt. $\angle = \angle EBC \therefore EB = EC \therefore AE^2 = 2EC^2$ (II. 11.) $= 2BE^2$.

10. See Example 6, Exercises xxii.

MENSURATION EXAMPLES. EXERCISES XXIV.

1. For Part 1, see Example 9, Exercises xxii. (2) $FG^2 = GC^2 + CF^2 = 2 \therefore FG = \sqrt{2} = EH$. $EF^2 = EB^2 + BF^2 = 8 \therefore EF = 2\sqrt{2} = HG \therefore$ the perimeter $= 6\sqrt{2} = 6(1{\cdot}41421) = 8{\cdot}49$ in. approx.

2. If the edge is straight, it will coincide with the line drawn. For Part 2, see Example 15, Exercises A.

3. The centre of the rolling circle is always at a distance of 3 in. from the centre of the fixed circle $\therefore$ the locus is a circle of rad. 3 in. centre at the centre of the fixed circle.

4. Draw a str. line AB 5 cms. long, and at A draw AD perp. to AB, making AD = 8 cms. Through D draw DCE $\parallel$ to AB, and bisect $\angle$BAD by AC meeting DCE at C. Thro. B draw BE $\parallel$ to AC to meet DCE at E. Altitude of parm. = 8 cms. $\therefore$ its area $= 8 \times AB = 40$ sq. cms., and $\angle BAC = \frac{1}{2}$ a rt. $\angle \therefore$ ABEC is the parm. reqd. By measurement AC = 11·3 cms.

5. Draw AB 10 cms. long and BC at rt. $\angle$s to it 4 cms. long. $AC^2 = AB^2 + BC^2 = 100 + 16$ cms. $\therefore AC = \sqrt{116} = 10{\cdot}77$ cms. $\therefore$ the dist. reqd. = 11 kilometres, to the nearest kilometre.

6. Let AB, CD be the chords, O the centre of the circle, OE perp. to AB, OF perp. to CD, so that $OE = \frac{3r}{5}$, and $OF = \frac{4r}{5}$, where r = the radius of the circle. $\angle OAB = \angle OBA$ (I. 5.) $\therefore$ from $\triangle$s OAE, OBE, AE = EB (I. 16.). Similarly, CF = FD $\therefore AB^2 = 4AE^2 = 4(AO^2 - EO^2) = 4\left(r^2 - \frac{9r^2}{25}\right) \therefore AB = \frac{8 \cdot r}{5}$. $CD^2 = 4CF^2 = 4(CO^2 - OF^2) = 4\left(r^2 - \frac{16r^2}{25}\right) \therefore CD = \frac{6r}{5} \therefore \frac{3 \cdot AB}{CD} = \frac{24}{6} = 4 \therefore$ the shorter chord is contained 4 times in 3 times the longer chord.

7. Draw AB, AC at rt. $\angle$s to one another and each equal to 2 in. By measurement $BC = 2{\cdot}83$ in.

8. Bisect AB by taking $AC = 3$ in. Draw CD at rt. $\angle$s to AB by I. 28. and cut off $CD = 3$ in. By measurement $AD = 4{\cdot}24$ in. Also by II. 11. $AD^2 = AC^2 + CD^2 = 18$ $\therefore$ $AD = 3\sqrt{2} = 3(1{\cdot}41421) = 4{\cdot}24$ in. approx.

9. Draw the $\triangle$ by the method of I. 25. Draw CD perp. to the base AB. By measurement, $CD = 2{\cdot}83$ in. Or, $\angle CAB = \angle CBA$ (I. 5.) $\therefore$ from $\triangle$s ADC, BDC, $AD = DB$ $\therefore$ $CD^2 = CA^2 - AD^2$ (II. 11.) $= 9 - 1 = 8$ $\therefore$ $CD = \sqrt{8} = 2{\cdot}83$ in. to the nearest hundredth of an inch.

10. Draw AB 4 in. long, and make $\angle BAC = 45°$, cut off $AC = 4$ in. Draw CD $\parallel$ to AB, and BD $\parallel$ to AC. Also draw CE perp. to AB. ABDC is the rhombus, and CE its altitude. By measurement, $CE = 2{\cdot}83$ in. $\therefore$ the area of the rhombus $= 4 \times 2{\cdot}83 = 11{\cdot}32$ sq. in. *N.B.*—We can only *measure* to the nearest hundredth of an inch. By calculation it will be found that $CE = 2{\cdot}82842$ more exactly. Whence the area $= 11{\cdot}31$ sq. in. to the nearest hundredth. The error in the measurement of CE is multiplied by 4 in finding the area.

11. By measurement the altitude of the $\triangle = 3{\cdot}46$ in. $\therefore$ the area $= \frac{1}{2} \times 4 \times 3{\cdot}46 = 6{\cdot}92$ sq. in. (This result is inaccurate for the reason given in the previous example.)

12. If x is the third side, $5^2 = x^2 + 4^2$ (II. 11.). Whence $x = 3$ in. $\therefore$ the area $= \frac{1}{2}$ alt. $\times$ base $= \frac{1}{2} \times 3 \times 4 = 6$ sq. in.

13. Let Ox, Oy be the perp^r str. lines. Mark points 2 units from Ox and 1 unit from Oy, 4 units from Ox and 2 units from Oy, and so on. The locus will then be seen to be a str. line thro. O.

14. Mark on the squared paper a rectangle whose sides are 3 units and 4 units long. We see then that the area of the rect. consists of 3 rows of square units, each row consisting of 4 sq. units $\therefore$ the area $= 12$ sq. units.

15. By I. 23. the locus is a str. line bisecting at rt. $\angle$s the str. line joining the given pts. To test the locus with circles, take centres on the locus.

16. Draw AB, BC at rt. $\angle$s to one another and each 3 cms. long. Draw AD at rt. $\angle$s to AB. With centre C and rad.

6 cms. describe a circle cutting AD in D, taking D on the same side of AB as the pt. C. ABCD is the reqd. quadl.

17. Draw AB 5 cms. long, and AC at rt. ∠s to it 4 cms. long. Draw CDE ∥ to AB. With centre A and rad. 5 cms. describe a circle cutting CDE at D. Draw BE ∥ to AD meeting CD at E. DABE is the reqd. rhombus.

MENSURATION PROBLEMS. EXERCISES XXV.

1. Area $=$ base $\times$ alt. $= 24 \times 2$ sq. in. $= 48$ sq. in.

2. Area $=$ breadth $\times$ length $= 18 \times 14\frac{1}{2}$ sq. ft. $= \frac{18}{9} \times \frac{29}{2}$ sq. yds. $= 29$ sq. yds.

3. Area $= \frac{1}{2}$ base $\times$ alt. $= 12 \times 14$ sq. in. $= 168$ sq. in.

4. Area $= 9^2 + 12^2$ (II. 11.) $= 3^2(3^2 + 4^2) = 15^2 = 225$ sq. in.

5. Let x be the side reqd. $x^2 + 48^2 = 52^2$ (II. 11.) $\therefore x^2 = (52 - 48)(52 + 48) = 4 \times 100 \therefore x = 20$ in.

6. Draw two str. lines AB, AC, each 2 in. long, at rt. ∠s to one another. $BC^2 = 8$ (II. 11.) $\therefore BC = \sqrt{8} = 2{\cdot}83$ in. by measurement.

7. Draw AB 2 cms. long, AC at rt. ∠s to it, 1 cm. long. $BC^2 = 4 + 1$ (II. 11.) $\therefore BC = \sqrt{5} = 2{\cdot}24$ cms. by measurement.

8. Draw AB 3 inches long, and BC at rt. ∠s to it. With centre A and rad. 4 inches describe a circle meeting BC at C. $BC^2 = AC^2 - AB^2$ (II. 11.) $= 16 - 9 = 7. \therefore BC = \sqrt{7} = 2{\cdot}65$ inches.

9. $AC^2 = 24^2 + 32^2$ (II. 11.) $= 8^2(3^2 + 4^2) = 8^2 \times 5^2 . \therefore AC = 40$ ft. $\therefore$ 10 ft. of the ladder projects beyond the top of the wall.

10. Draw AB, BC at rt. ∠s to one another making $AB = BC = 5$ cms. $AC^2 = AB^2 + BC^2$ (II. 11.) $= 2 . AB^2$. By measurement, $AC = 7{\cdot}07$ cms.

11. If AB represents 72 ft., AC 30 ft., then BC represents 78 ft. Also $BC^2 - AB^2 = 78^2 - 72^2 = 6 \times 150 = 30^2 = AC^2 \therefore \angle CAB$ is a rt. ∠. (II. 12.).

12. 4 in. $= 10{\cdot}2$ cms. nearly.

13. 6 cms. $= 2{\cdot}36$ in.

14. If ABC be the △ such that $AB = 6$ cms. $\angle ABC = 60°$, $\angle ACB =$ a rt. ∠, produce BC to D making $CD = CB$. Join AD.

From $\triangle$s ACD, ACB, AD = AB, and $\angle$ADC = $\angle$ABC = 60° (I. 4.) $\therefore$ $\angle$BAD = 60° (I. 22.) $\therefore$ $\triangle$ABD is equilateral (I. 6.) and BC = $\frac{1}{2}$BD = $\frac{1}{2}$BA = 3 cms. By measurement, AC = 5·2 cms.

15. Let ABCD be the quadl. such that AC = 30, and BD = 40 ft. Let AC, BD cut at rt. $\angle$s at O. Area of figure = $\triangle$ABD + $\triangle$BCD = $\frac{1}{2}$AO . BD + $\frac{1}{2}$CO . BD = $\frac{1}{2}$(AO + CO) BD = $\frac{1}{2}$. 30 × 40 = 600 sq. ft.

16. Let ABC be the $\triangle$ such that AB = 4 in., BC = 6 in., and $\angle$ABC = 30°. Draw AD perp. to BC. By measurement, or as in Example 14 above, AD = 2 in. $\therefore$ area of $\triangle$ = $\frac{1}{2}$AD . BC = 6 sq. in.

17. If x be a side of the square, $x^2 = 15$ acres = 150 sq. chains $\therefore$ $x = \sqrt{150}$ chains = 12·25 chains approx. = 12 chains 25 links.

18. If x be a side of the sq. then $2x^2 = 36$ (II. 11.) $\therefore$ area of square = x^2 sq. ft. = 18 sq. ft.

19. If x ft. be the length of the other side of the rect. $x^2 = 8^2 - 5^2 = 39$ sq. ft. $\therefore$ area of rect. = $5 \times \sqrt{39} = 5 \times 6·245$ = 31 sq. ft. (to the nearest sq. ft.).

20. Area of hall = 8 × 12 × 144 sq. in. Area of each tile = 9 sq. in. $\therefore$ no. of tiles = $\frac{8 \times 12 \times 144}{9} = 1536$.

21. Let x ft. be the length of the hall. $8x$ = area of hall in sq. ft. = $\frac{1920 \times 16}{144} = \frac{640}{3}$ ft. $\therefore$ $x = \frac{80}{3}$ ft. = 26 ft. 8 in.

22. Let x yds. be the length of the other side. $\frac{1}{2}x \times 22$ = area of $\triangle$ = 1210 $\therefore$ $x = 110$ yds.

23. Let x ft. be the length of the other side. $x^2 = 39^2 - 15^2$ (II. 11.) $= 3^2(13^2 - 5^2) = 3^2(13-5)(13+5)$, $3^2 \times 144$ $\therefore$ $x = 36$ ft. $\therefore$ area of $\triangle = \frac{1}{2}x \times 15 = 270$ sq. ft.

24. Draw AD perp. to BC. $\angle$CAD = 45° (I. 22.) $\therefore$ $2\text{AD}^2 = \text{AC}^2$ (II. 11.) = 25. $\therefore$ AD = $\frac{5\sqrt{2}}{2} = \frac{5}{2}(1·41421)$ = 3·54 ft. approx. Area of $\triangle$ = $\frac{1}{2}$AD × BC = $\frac{1}{2} \times \frac{5\sqrt{2}}{2} \times 12 = 15\sqrt{2}$ sq. ft. = 21·21 sq. ft. approx.

25. Let x ft. be the reqd. length of the perp. $\frac{1}{2}x \times 14$ = area of $\triangle = \frac{1}{2} \times 6 \times 12$ $\therefore$ $x = \frac{6 \times 12}{14} = 5\frac{1}{7}$ ft.

26. Construct the $\triangle$ as in I. 25. By measurement, the perp. on the 5 cm. side = 5·4 cms. Area = $\frac{1}{2} \times 5·4 \times 5 = 13·5$ sq. cms.

27. Let ABCD be the rhombus having $\angle ADC = 45°$. Draw AE perp. to CD. $\angle DAE = 45°$ (I. 22.) $\therefore$ AE = DE (I. 6.) $\therefore 2AE^2 = AD^2 = 36$, and $AE = 3\sqrt{2}$ ft. Area of rhombus $= 6 \times 3\sqrt{2}$ sq. ft. $= 18(1{\cdot}41421) = 25$ sq. ft. (to the nearest sq. ft.).

28. If x ft. be the length of the diagonal $x^2 = 2 \times 18^2$ (II. 11.) $\therefore x = 18\sqrt{2} = 25{\cdot}46$ ft. to the nearest hundredth of a foot.

29. Dist. reqd. $= \sqrt{30^2 + 40^2}$. (II. 11.) $= 50$ miles.

30. If x links be the length of the diagonal, $x^2 = 2 \times 410^2$ (II 11.). $x = 410\sqrt{2} = 410(1{\cdot}41421) = 580$ links (to the nearest link) $= 5$ chains 80 links.

31. If x ft. be the dist. reqd. $x^2 = 31^2 - 23^2$ (II. 11.) $= (31 - 23)(31 + 23) = 8 \times 54$ $\therefore x = 12\sqrt{3}$ ft. $= 20{\cdot}78$ ft. (to two dec. places).

32. Let AB ($= x$. ft.) be the height of the tower, BC its shadow. As in Example 14 above, AC = 2AB = $2x$ ft. $\therefore 4x^2 = x^2 + 200^2$ (II. 11.). $3x^2 = 200^2$, $x = \frac{200}{3}\sqrt{3} = 115{\cdot}47$ ft. (to two dec. places).

33. Draw ABC horizontally so that AB = BC = 2 inches. Draw BD, 1 in. long, perp. to ABC, to represent the boy; and CE also perp. to ABC. Produce AD to meet CE at E. CE is the lamp-post. Draw DF $\parallel$ to BC to meet DE at F. FC = BD (II. 2.) 1 in. and DF = BC = AB. From $\triangle$s EFD, DBA, EF = BD = 1 in. $\therefore$ EC = 2 in. $\therefore$ the post is 10 ft. high.

34. Let x be the hypotenuse, then $x^2 = (m^2 - n^2)^2 + (2mn)^2 = m^4 + 2m^2n^2 + n^4$ $\therefore x = m^2 + n^2$ cms. When $m = n + 1$, $m^2 + n^2 - 2mn = \overline{m - n}|^2 = 1$. When $m = 13$ and $n = 12$, the sides are respectively 25, 312, 313 cms.

35. Let ABCD be the trapezium, such that AB = 9, BC = 10, CD = 30, DA = 17 ft. Let x be the perp. dist. between AB and CD. Draw AE, BF perp. to CD. DE + EF + CF = 30 $\therefore$ $\sqrt{17^2 - x^2} + 9 + \sqrt{100 - x^2} = 30$ (II. 11. and 2.) $\therefore \sqrt{17^2 - x^2} = 21 - \sqrt{100 - x^2}$, $289 - x^2 = 441 + 100 - x^2 - 42\sqrt{100 - x^2}$, whence $\sqrt{100 - x^2} = 6$ and $x = 8$ ft. $\therefore$ area of trapezium $= \frac{1}{2}(9 + 30)8 = 156$ sq. ft.

36. Make DQ equal to BP. Join AQ, BD. DQ = BP $\therefore \triangle$ADQ = $\triangle$PBQ. CQ = AP $\therefore \triangle$APQ = $\triangle$BQC $\therefore$ fig. ADQP $= \frac{1}{2}$ parm. ABCD.

37. (1) Let ABCD be the sq., E the mid. point of BC. Area of $\triangle$DCE $= \frac{1}{2}$DC . CE $= 9$ sq. in. Area of ABED $= 36 - 9 = 27$ sq. in. (2) Join AC cutting DE at O. Take F the mid. pt. of AD and join FB, cutting AC at P. FP is $\parallel$ to DO (II. I.) and AF = FD $\therefore$ AP = PO. Similarly CO = OP $\therefore$ AO = 2OC $\therefore$ $\triangle$AOD $= 2\triangle$DOC, *i.e.* $\triangle$DOC $= \frac{1}{3}\triangle$ADC $= \frac{1}{6}$ of sq. $= 6$ sq. in. $\therefore$ $\triangle$AOD $= 2\triangle$DOC $= 12$ sq. in. and $\triangle$COE $= \triangle$DCE $- \triangle$DOC $= 3$ sq. in. and the remaining part AOEB $= 15$ sq. in.

38. Let ABCD be the trapezium, such that AB $= 20$, BC $= 13$, CD $= 34$, DA $= 15$ yds. Draw AE, and BF perp. to CD, and let x yds. be the perp. dist. between AB and CD, so that AE = BF $= x$. DE + EF + CF $= 34$ $\therefore$ $\sqrt{15^2 - x^2} + 20 + \sqrt{13^2 - x^2} = 34$ (II. 11. and 2.) $\therefore$ $\sqrt{15^2 - x^2} = 14 - \sqrt{13^2 - x^2}$, $225 - x^2 = 196 - 28\sqrt{13^2 - x^2} + 169 - x^2$ $\therefore$ $28\sqrt{13^2 - x^2} = 140$ $\therefore$ $169 - x^2 = 25$ $\therefore$ $x = 12$ $\therefore$ area of trapezium $= \frac{1}{2}$(AB + CD)AE $= \frac{1}{2} . 54 \times 12 = 324$ sq. yds.

*** **The additional Exercises, 39-81, will be found on pages 174-177.**

MISCELLANEOUS EXERCISES XXVI.

1. $a + b > c$ in all $\triangle$s (I. 12.) $a + b = c$ is untrue for all $\triangle$s, for the same reason. $a + b < c$ is untrue for all $\triangle$s, for the same reason. $a^2 + b^2 = c^2$ is true for right $\angle$d $\triangle$s, when c is the side opp. the rt. $\angle$ (II. 11.).

2. If ABCD is the reqd. quadl. and E the mid. pt. of BC, DE is perp. to BC for DB = DC. (I. 7.). Thro. D draw FDH $\parallel$ to BC, forming the rectangle BCHF. $\angle$ABD = complement of $\angle$DBC = $\angle$ACB $\because$ from $\triangle$s ABC, DFB, BA = FD (I. 16.) = BE $= \frac{1}{2}$BC. Hence the following construction. Draw any str. line BC, and BA at rt. $\angle$s to it, making BA $= \frac{1}{2}$BC. Join AC. Draw BO perp. to AC and produce it to D making BD = CA. ABCD is the reqd. quadl.

3. Let AB be gr. than AC, so that $\angle$ACB $>$ $\angle$ABC $\therefore$ $\angle$BAF, the complement of $\angle$ABC $>$ $\angle$CAF the complement of $\angle$ACB $\therefore$ $\angle$BAF $> \frac{1}{2}\angle$BAC $\therefore$ D falls between B and F. Perps. from D to AB, AC are equal (I. 24.) $\therefore$ $\triangle$ABD $>$ $\triangle$ADC, *i.e.* $> \frac{1}{2}\triangle$ABC $\therefore$ BD $>$ BE $\therefore$ E the mid. pt. of BC lies between B and D $\therefore$ AE, AD, AF are in order of magnitude (Exercises xviii. 3.).

4. Let ABC be the equilateral $\triangle$, BL, CN the altitudes of the parms. on AB and AC. Draw BK perp. to BC and equal to

BL + CN. Draw EKF ∥ to BC, and any ∥s BE, CF completing the parm. BEFC. BEFC is the parm. reqd. (II. 4.).

5. Let ABC be the △, DE the given perimeter. From DE cut off DF = BC. Bisect FE at G. Bisect BC at L. Thro. A draw AKH ∥ to BC. With centre L and rad. FG describe a circle cutting AKH at K. Join LK and complete the parm. BLKH. The perimeter of this parm. = 2 . BL + 2 . KL = DE. Also parm. BL = 2△ABL = △ABC ∴ HBLK is the parm. reqd.

6. Let A be the given pt., BC the given line. Draw AD perp. to BC, and produce it to E making DE = DA. With centre D and rad. DA or DE describe a circle cutting BC at F and G. AFEG is the reqd sq. ∠GAD = $\frac{1}{2}$ a rt. ∠ = ∠DEF (I. 22.) ∴ AG is ∥ to FE. Similarly AF is ∥ to GE ∴ AFEG is a parm. Also AG = AF (I. 4.) and ∠GAF = a rt. ∠ ∴ AFEG is a sq.

7. Let △ADE be described on the same side of AD as the pt. B. ∠EAD = $\frac{2}{3}$ of a rt. ∠ (I. 22.) = ∠BAC ∴ ∠EAB = ∠DAC ∴ from △s EAB, DAC, EB = CD (I. 4.). If △ADE be described on the opp. side of AD to the pt. B, CE will be equal to BD.

8. Let M and N be on the same side of PQ. Draw MC, NC to meet on PQ at C, so that ∠BCP = ∠NCQ (Exercises xviii., Example 2). Thro. L draw BLA ∥ to NC to meet CM at B, and PQ at A. Thro. A draw AD ∥ to CB to meet NC at D. ∠BAC = ∠NCQ (I. 20.) = ∠BCA ∴ BA = BC. Also, by construction, ABCD is a parm. ∴ CD = BA = BC = AD ∴ ABCD is also a rhombus, and the fig. reqd.

9. Produce AB to E making BE equal to the given line. Join DE, and draw DX making ∠EDX = ∠DEB. Draw BF ∥ to DE to meet DX in F. ∠XFB = ∠XDE (I. 20.) = ∠XED = ∠XBF (I. 20) ∴ BX = FX (I. 6.). Also XD = XE (I. 6.) ∴ DF = BE ∴ DX − BX = DX − FX = DF = BE ∴ X is the pt. reqd.

10. Let DPE meet AB in D and AC produced in E. Draw CF ∥ to DB to meet PE at F. ∠PCE > ∠ABC (I. 8.) > ∠PCF (I. 20.) ∴ F falls between P and E. △DPB = △FPC (I. 3. 20. 16.) ∴ adding fig. ADPC to each △ABC = fig. DFCA < △ADE.

11. Let △ABC have AB > AC. With centre A and rad. equal to $\frac{1}{2}$(AB + AC) describe a circle cutting BC at P. 2AP = AB + AC ∴ AP − AC = AB − AP ∴ P is the pt. reqd.

12. Let the lines EP, FP bisecting AD and BC at rt. ∠s meet at P. From △s AEP, DEP, PA = PD (I. 4.). From △s BFP, CFP, BP = CP (I. 4.) ∴ △s APB, DPC are equal in all respects (I. 7.). By bisecting AC and BD at rt. ∠s, another pt. Q may be found satisfying the reqd. conditions.

13. Draw AD perp. to BC in the equilateral △ABC. BD = DC (I. 16.) ∴ AB = 2BD. $BD^2 + AD^2 = AB^2$ (II. 11.) $= 4BD^2$ ∴ $AD^2 = 3BD^2$.

14. Let ABCD be the given sq. P the given pt. in AB, AP being gr. than PB. Join BD. △ABD = $\frac{1}{2}$ the sq. Draw PF as in Exercises xxiii. 1, so that PF bisects the △ADB. △APF = $\frac{1}{2}$△ABD = $\frac{1}{4}$ of the sq. In DC make DG = BP. Join PG. By superposition fig. APGD = fig. PBCG = $\frac{1}{2}$ sq. ABCD ∴ since △APF = $\frac{1}{4}$ of the sq., fig. FPGD = $\frac{1}{4}$ of the sq. From GC (> GD) cut off GH = $\frac{1}{2}$DC. △PGH = $\frac{1}{4}$ of the sq. (II. 9.) ∴ PF, PG, PH are the reqd. lines.

15. In △ABC let AD be drawn perp. to BC, and take any pt. P in BC. $PB^2 = PD^2 + BD^2$ (II. 11.). $PC^2 = PD^2 + CD^2$ (II. 11.) ∴ $PB^2 \backsim PC^2 = BD^2 \backsim CD^2$.

15 (2). $PB^2 - PC^2 = BD^2 - CD^2 = BA^2 - CA^2$. This is only true when P lies in AD, as may be seen by drawing a perp. QH to BC from a point Q outside AD. ($BQ^2 - CQ^2 = BH^2 - CH^2$ which is not equal to $BD^2 - CD^2$.) Let P be the intersection of the altitudes AD, BE. Then $PB^2 - PC^2 = AB^2 - AC^2$, and $PA^2 - PC^2 = AB^2 - BC^2$ ∴ by subtraction $PB^2 - PA^2 = BC^2 - AC^2$ ∴ P must lie in the altitude CF.

16. CA = CD ∴ ∠CAD = ∠CDA. CA = CE ∴ ∠CAE = ∠CEA ∴ ∠DAE = ∠ADE + ∠AED ∴ ∠DAE = a rt. ∠ (I. 22.).

17. If $x°$ be the smallest angle. $6x = 180$ (I. 22.) ∴ $x = 30°$ ∴ we have to describe a △ whose angles are 30°, 60°, 90°. Describe an equilateral △ABC, and draw AD perp. to BC. △s ADB, ADC both satisfy the reqd. conditions.

18. Join EC, EB. △FCB = △FCA (II. 5.) = △ECA (II. 5.) = △EBA (II. 5.).

19. Reduce the given fig. to a △ABC as in (II. 15.). Bisect BC at D, and BD at E. Draw EF perp. to BD to meet AF ∥ to BC at F. Produce FE to G making EG = EF. BFDG is the reqd. rhombus. For △BFD = $\frac{1}{2}$△ABC (II. 6.). Also BF = FD

from $\triangle$s BEF, DEF (I. 4.). Similarly BG = DG, and $\triangle$BFD = $\triangle$BGD (II. 5.). Also BF = DG from $\triangle$s BEF, DEG (I. 4.). Similarly FD = BG.

20. Let ABC be the $\triangle$ so that $\angle$ACB = 2$\angle$ABC, and let AD be perp. to BC. Make $\angle$ABF equal to $\angle$ABC, and draw AF $\parallel$ to BC to meet BF at F. Draw FH perp. to BC. From $\triangle$s BHF, CDA, BF = CA, and BH = CD. Also $\angle$FAB = $\angle$ABC (I. 20.) = $\angle$FBA $\therefore$ FA = FB (I. 6.) $\therefore$ BD − CD = BD − BH = HD = AF = BF = AC.

21. Let ABCD, PQRS be the quadls., E being the mid. pt. of AB and PQ, F of BC and QR, G of CD and RS, H of DA and SP. Also let P, Q, R fall without ABCD, and S within. Join AP, BQ, CR, SD. Then by I. 3. and 4. $\triangle$AHP = $\triangle$SHD. $\triangle$BEQ = $\triangle$AEP, $\triangle$BFQ = $\triangle$CFR, $\triangle$CGR = $\triangle$SGD. But fig. PAEQFCRGDH = ABCD + $\triangle$AHP + $\triangle$QBE + $\triangle$QBF + $\triangle$CRG and it also = PQRS + $\triangle$EAP + $\triangle$SHD + $\triangle$GSD + $\triangle$FCR $\therefore$ ABCD = PQRS.

Another method. Let ABCD be a quadl. having E, F, G, H the mid. pts. of AB, BC, CD, DA. $\triangle$AEH = $\frac{1}{4}$$\triangle$ABD, $\triangle$CGF = $\frac{1}{4}$$\triangle$CDB (Exercises XX. 1, 2.) $\therefore$ $\triangle$AEH + $\triangle$CGF = $\frac{1}{4}$ABCD. Similarly $\triangle$BFE + $\triangle$DHG = $\frac{1}{4}$ABCD $\therefore$ the $\triangle$s exterior to the parm. EFGH = $\frac{1}{2}$ABCD $\therefore$ ABCD = twice parm. EFGH. Similarly any other quadl. PQRS which has E, F, G, H for the mid. pts. of its sides = twice parm. EFGH $\therefore$ ABCD = PQRS.

22. Produce QP to S making PS = PQ. Join PS. $QR^2 = QP^2 + PR^2 = 4QP^2$ (II. 11.) $\therefore$ QR = 2 . PQ. From $\triangle$s RPQ, RPS, SR = QR, SP = QP and $\angle$QRP = $\angle$SRP (I. 4.) $\therefore$ QRS is an equilateral $\triangle$ and $\angle$RQP = 60° = 2$\angle$QRP.

23. Take X in DB nearer to B than D; and let FXHB, GXED by the parms. about DB, F lying in AB, G in AD, E in CD, H in CB. $\frac{1}{2}$ parm. ABCD + $\triangle$ACX = fig. ADCX = parm. GE + $\triangle$AGX + $\triangle$ECX = parm. GE + complement AX (1) $\therefore$ parm. GE + complement AX − $\triangle$ACX = $\frac{1}{2}$ parm. ABCD. Also $\frac{1}{2}$ parm. ABCD = $\triangle$ACB = parm. FH + $\triangle$ACX + $\triangle$AFX + $\triangle$CHX = parm. FH + $\triangle$ACX + complement AX $\therefore$ from (1) parm. GE − parm. FH = 2$\triangle$ACX.

24. Let ABCD be the quadl. $\triangle$ABC = $\frac{1}{2}$ quadl. = $\triangle$DBC $\therefore$ AD is $\parallel$ to BC (II. 7.). Similarly AB is $\parallel$ to DC $\therefore$ ABCD is a parm.

25. Thro. E draw FEG $\parallel$ to AB to meet AD at F and BC at G. $\triangle$DEF = $\triangle$CEG (I. 20. and 16.) $\therefore$ parm. ABGF = fig. ABCD. Also $\triangle$AEB = $\frac{1}{2}$ parm. ABGF (II. 9.) = $\frac{1}{2}$ fig. ABCD.

26. Let ABCD be the larger sq., AEFG the smaller, BAE being a str. line, and G lying in AD. From AB cut off AH = GD. Join FH and cut along this line. Join CH and cut along this line also. Place $\triangle$HBC so that H coincides with F and HB with FG (HB = FG) and BC falls along GD, $\triangle$HBC occupying the position FGK. DK = GK − GD = CB − BH = AB − AH = HB = FE $\therefore$ $\triangle$KDC = $\triangle$FEH in all respects. Also $FK^2 = HB^2 + BC^2 = FE^2 + EH^2 = FH^2$ $\therefore$ FK = FH. $\angle$FHE = complement of $\angle$HFE = complement of $\angle$CHB $\therefore$ FHC is a rt. $\angle$. In like manner $\angle$FKC is a rt. $\angle$ and $\angle$KCH $\therefore$ KFHC is a sq. equal to the two given squares.

27. Let EFKB HDGF be parms. about BD a diagonal of parm. ABCD, E lying in AB, K in BC, G in CD, H in DA. Join HE, GK. $\triangle$HEG = $\frac{1}{2}$ parm. ADGE (II. 9.) = $\frac{1}{2}$ parm. HDCK (II. 10.) = $\triangle$HKG (II. 9.) $\therefore$ EK is $\parallel$ to HG (II. 7.).

28. Let ABCD be the rect., O the pt. within it. Draw MON perp. to AD and BC, meeting AD at N and BC at M. $AO^2 + OC^2 = (AN^2 + ON^2) + (OM^2 + CM^2)$ (II. 11.) $= BM^2 + OM^2 + ON^2 + DN^2$ (II. 2.) $= OB^2 + OD^2$ (II. 11.).

29. Let AB be > AC. Draw BH and CF perp. to DME. From $\triangle$s BMH, CMF, BH = CF (I. 3. and 16.). Also $\angle$HDB = $\angle$ADE = $\angle$AED (I. 6.) $\therefore$ from $\triangle$s HDB, EFC, BD = CE (I. 16.).

30. Let ABCD be the sq. and let the given side of the rect. be > AB. Produce AD to E, so that AE = the given side. Draw EHF $\parallel$ to AB to meet BC produced in F. Join AF meeting CD at G. Draw HGK $\parallel$ to EA or FB. Rect. EAKH = DK + complement EG = DK + complement GB (II. 10.) = DB = sq. ABCD. The same construction holds when AE < AB.

31. Let ABCD be the parm. E the mid. pt. of BC, F the mid. pt. of DA. Let ED and BF meet AC at G and H. Draw HK $\parallel$ to AD or BC to meet DE at K. HK = FD (II. 2.) = AF $\therefore$ from $\triangle$s HKG, AFH, AH = HG (I. 20. and 16.). Also from $\triangle$s HGK CGE, HG = GC (I. 20. and 16.) $\therefore$ AC is trisected at G and H.

32. Take O any pt. within the $\triangle$ABC. OB + OC < AB + AC, OA + OB < CA + CB, OC + OA < BC + BA (I. 13.) $\therefore$ adding, 2(OA + OB + OC) < 2(AB + BC + CA) $\therefore$ OA + OB + OC < AB + BC + CA. Also OB + OC > BC, OC + OA > AC, OA + OB > AB (I. 12.) $\therefore$ add-

ing $2(OA+OB+OC)>AB+BC+CA$, *i.e.* $OA+OB+OC>\frac{1}{2}(AB+BC+CA)$.

33. Let AB, AC be the given lines. In AB take DE equal to the given length. Draw DF perp. to AB, and EF perp. to AC, these lines meeting at F. Draw FG || to AB to meet AC at G. Draw GK perp. to AB, and GH perp. to AC to meet AB at H. In $\triangle$s GKH, FDE, $GK=FD$ (II. 2.). $\angle GKH=\angle FDE$ and $\angle GHK=\angle FED$ (I. 20.) $\therefore KH=DE$ $\therefore$ GK, GH are the reqd. lines.

34. Let ABCD be the given sq. on AB, on the same side of it as the pts. C, D, describe an equilateral $\triangle ABE$. Bisect $\angle EAB$ by AF, meeting BC at F. $\angle FAB=\frac{1}{2}\angle EAB=30°$ (I. 22.) $\therefore \angle AFB=60°$ $\therefore$ AFB is half the equilateral $\triangle$ on AF $\therefore FB=\frac{1}{2}AF$ $\therefore AB^2=AF^2-FB^2$ (II. 11.) $=3FB^2$ $\therefore$ the sq. on FB is the sq. reqd.

35. Draw DE perp. to AB. $\angle EBD=\frac{1}{2}$ a rt. $\angle$ $\therefore \angle BDE=\frac{1}{2}$ a rt. $\angle$ (I. 22.) $\therefore ED=EB$ (I. 6.). From $\triangle$s ADE, ADC, $DE=DC$ and $AE=AC$ (I. 26.) $\therefore AB=AE+EB=AC+ED=AC+CD$ $\therefore CD=AB-AC$.

36. Let AB, AC be the given str. lines, and BD || to the third given line. Cut off BD equal to the given length. Draw DE || to AB to meet AC at E. Draw EF || to DB to meet AB at F. $EF=DB=$ the given length; and it is in the reqd. direction. The problem is impossible (1) when the given direction is || to AB or AC; (2) when DE does not meet AC, *i.e.* when AB and AC are ||.

37. Let AC be $<$ AB, so that $\angle ACB>\angle ABC$ (I. 10.). Produce BA to D and let AE bisect $\angle DAC$. Draw AF || to BC, so that F falls within $\angle DAC$. $\angle DAC=\angle B+\angle C>2\angle B$ $\therefore \angle DAE>\angle B>\angle DAF$ (I. 20.) $\therefore$ AE meets BC produced beyond C. Similarly if $AC>AB$, AE will meet CB produced beyond B.

38. With the fig. of II. 10, let parm. FH = parm. KG. Since the complements are equal, $GK+GH=\frac{1}{2}$ parm. ABCD, *i.e.* parm. HBCK $=\frac{1}{2}$ parm. ABCD = parm. AHKD $\therefore AH=HB$ $\therefore$ parm. HG = parm. FH = parm. GK = parm. FK.

39. Let ABCD be the given parm. Produce AB to E making $BE=AB$. Make $\angle EAK$ equal to the given $\angle$ and let CD meet AK at K. Join KE. $\triangle AKE=2\triangle ABK$ (II 6.) = parm. ABCD (II. 9.) and has $\angle KAE$ equal to the given angle.

40. Let ABCD be the given rect. AB being > AD. On AB describe the sq. ABFE, AE passing thro. D, and BF thro. C. Bisect AE at O and with centre O and rad. OA or OE describe a circle to meet CD produced at H. Join AH, HE, HO. ∠OAH = ∠OHA, and ∠OEH = ∠OHE ∴ ∠AHE = ∠HAE + ∠HEA ∴ ∠AHE = a rt. ∠ (I. 22.) ∴ as in (II. 11) the sq. on AH = rect. ABCD.

41. If ABCD is the given parm. draw AF, BE perp. to CD meeting it in E and F. Rect. ABEF = parm. ABCD (II. 3.). Then use the preceding exercise.

42. Let ABCD be a parm. and O a pt. within it. Let EOF be ∥ to AD or CB, and GOH ∥ to CD or AB. Also let parm. DO = parm. BO. Join AO, OC. △AGO = △AFO and △EOC = △OHC. Also parm. DO = parm. BO ∴ △AGO + parm. DO + △EOC = $\frac{1}{2}$ parm. ABCD = △DAC ∴ AOC must be a str. line.

43. Draw EPF ∥ to AB and CD meeting AD at E, and BC at F. Draw GPH ∥ to AD and BC meeting AB at G and CD at H. △APB = △AGP + △PGB = $\frac{1}{2}$ parm. EG + $\frac{1}{2}$ parm. GF (II. 9.) = $\frac{1}{2}$ parm. HF + $\frac{1}{2}$ parm. GF (II. 10.) = △CPF + △FPB (II. 9.) = △CPB.

44. Let ABCD be the given parm. It is reqd. to describe a parm. equal to ABCD, having an angle equal to ∠DAB, and one side equal to the given line P. Let P be > AD. From AD produced cut off AE = P. Complete parm. BAEF. Join AF cutting CD at G. Draw HGK ∥ to AE or BF meeting EF at H, and AB at K. Parm. EK = parm. EG + parm. DK = parm. GB + parm. DK (II. 10.) = parm. ABCD ∴ EAKH is the reqd. parm.

45. Let ABCD be the given quadl. Join DB. Draw EAF, GCK ∥ to DB, and any two parallels thro. D and B to form the parm. EGKF. EGKF = EDBF + DGKB = 2△ADB + 2△DCB (II. 9.) = 2 fig. ABCD.

46. Let ABC be the given △. Bisect BC at D and join AD. Draw AE ∥ to BC and BE ∥ to AD to meet at E. With centre B and rad. equal to $\frac{1}{2}$(AB + AC) describe a circle cutting AE at H. Draw DK ∥ to BH. Parm. HBDK = parm. EBDA (II. 3.) = 2△ABD (II. 9.) = △ABC (II. 6.). Also its perimeter = 2 HB + 2BD = AB + AC + BC ∴ HBDK is the reqd. parm.

47. With the fig. of II. 10. take P the mid. pt. of AC and drawn MPN ∥ to AB and CD, meeting AD at M and BC at N.

Also draw RPS ‖ to AD and BC, meeting AB at R, and CD at S. Let HEK meet MP at L, E lying between A and P, L lying between P and M But PM = PN ∴ PN > LM ∴ parm. PG > parm. EM. In parm. ARPM, complement ER = complement EM ∴ adding RG to each, parm. HG = parm RG + parm. EM < parm. RG + parm. PG < parm RBNP. Thus we see that the complement HG is greatest when E lies at P the mid. pt. of AC.

48. Join GD, BF. △GDB = $\frac{1}{2}$ rect. ABDE (II. 9.) = $\frac{1}{2}$ rect. ACFG = △GBF (II. 9.) ∴ DF is ‖ to GB (II. 7).

49. Let ABCD be the quadl. Join BD. Of the △s ABD, ACD let ABD be the smaller Draw AE ‖ to BD to meet CB produced in E. Join DE. Bisect CE at F and join DF. △EBD = △ABD (II. 5.) ∴ adding BCD to each, △EDC = quadl. ABCD ∴ △DFC = $\frac{1}{2}$△EDC (II. 6.) = $\frac{1}{2}$ quadl. ABCD ∴ DF bisects the quadl.

50. Produce GF to meet CB at H. Join AH, meeting EF at K. Draw LKM ‖ to GF or AEB cutting AG at L, and CH at M. Parm. KB = parm. GK (II. 10) ∴ adding LE to each, parm. LB = parm. GE ∴ rect. LDCM = rect. ABCD + rect. AEFG.

51. △FHG + △FKG = $\frac{1}{2}$ parm. AG + $\frac{1}{2}$ parm. CF (II. 9.) = $\frac{1}{2}$ parm. ABCD = △ABC (II. 2.)

52. Let ABC be the given △. Bisect BC at D, and draw DE, CF perp. to BDC to meet EFH, ‖ to BC, at E and F. Along DC make DG equal to the given str. line, and draw GH perp. to DG. Join DH, cutting CF at K. Draw NKM ‖ to EF or BC, meeting DE at N and GH at M. Rect. NDGM = rect. NC + rect. CM = rect. NC + rect. EK (II. 10.) = rect. EDCF = 2△ADC (II. 9.) = △ABC.

53. Let ABCD be the given parm. Draw EF ‖ to AB and at the given perpendicular distance from AB. Let EF cut AD at E. Join BE and produce it to meet CD produced at G. Draw GHK ‖ to AD or BC to meet FE and BA produced at H and K. Parm. KBFH = parm. AF + parm KE = parm. AF + parm. EC (II. 10.) = parm. ABCD. Also KBEH is equiangular to ABCD, and is therefore the parm. reqd.

54. Let ABCD be the given quadl. Join AC, and draw DE ‖ to AC to meet BA produced at E. Join EC. Draw EF ‖ to BC, and CF ‖ to EB to meet at F. Bisect BE at G and draw GH

|| to BC to meet CF at H. Parm. GBCH = $\frac{1}{2}$ parm. EBCF (II. 4.) = △EBC (II. 2.) = △ABC + △EAC = △ABC + △ACD (II. 5.) = quadl. ABCD.

55. Draw AHK perp. to GB, meeting GB at H and EC at K. In △s GAB, CAE, AG = AC, AB = AE, ∠ at A is common ∴ ∠ACE = ∠AGB (I. 4.) = complement of ∠GAH (I. 22.) = ∠KAC ∴ CK = AK (I. 6.). Also ∠KEA = complement of ∠KCA (I. 22.) = complement of ∠KAC = ∠KAE ∴ KE = KA (I. 6.) ∴ KE = KC.

56. Let ABC be the △ such that AB = 2 . AC. Bisect AB at D and draw DE perp. to AB to meet BC at E. Join AE. From △s ADE, BDE, ∠EAD = ∠EBD (I. 4) ∴ ∠AEC = 2∠ABE (I. 22.). Also in the rt. ∠d. △ AED, AE > AD ∴ AE > AC ∴ ∠ACE > ∠AEC, *i.e.* ∠ACB > 2∠ABC.

57. Let ABC be an isos. △ having AB = AC, and ∠BAC = 30°. On the same side of AC as the pt. B describe an equilateral △AEC. Let EC meet AB in F ∠EAF = 30°. From △s AFE, AFC, AF is perp. to EC and bisects it (I. 4.). In the rt. ∠d. △BFC, BC > CF, *i.e.* BC > $\frac{1}{2}$CE, *i.e.* BC > $\frac{1}{2}$AC.

58. Let PB produced meet QR at S, and QB produced meet CP at T. ∠QBR = a rt. ∠ ∴ ∠TBR = a rt. ∠ ∴ ∠TBC = $\frac{1}{2}$ a rt. ∠ = ∠TCB ∴ BT is perp. to CT (I. 22.) ∴ from △s BTC, BRC, BT = BR. Also AQTP is a rectangle ∴ PT = AQ = QB ∴ in △s PTB, QBR, BT = BR, PT = QB and ∠PTB = a rt. ∠ = ∠QBR ∴ PB = QR and ∠BPT = ∠BQR (I. 4.) ∴ in △s PTB, QSB, ∠BPT = ∠BQS, ∠PBT = ∠QBS (I. 3) ∴ ∠BTP = ∠BSQ ∴ ∠BSQ = a rt ∠, *i.e.* PB is at rt. ∠s to QR.

59. Let AC meet HK at E. ∠AHB = ∠B (I. 5.) = ∠D = ∠AKD (I. 5.). Also AB = AD ∴ from △s ABH, ADK, BH = DK (I. 16.) ∴ CH = CK ∴ ∠KHC = 90° − $\frac{1}{2}$∠C ∴ ∠B + 60° + 90° − $\frac{1}{2}$∠C = ∠AHB + ∠AHK + ∠KHC = 180° ∴ ∠B − $\frac{1}{2}$∠C = 30°. But ∠B + ∠C = 180° ∴ $\frac{3}{2}$∠C = 150° ∴ ∠C = 100° = $\frac{10}{9}$ of a rt ∠.

EXERCISES XXVII.

1. Let AB, CD be equal chords, E the centre. ∠AEB = ∠CED by I. 7.

2. If AB > CD, ∠AEB > ∠CED by I. 15.

3. If ABC are pts. of intersection of str. line and circle, D the centre, $\angle$DAB $=$ $\angle$DCB (I. 5.) $=$ $\angle$DBC (I. 5.), which is impossible by I. 8.

4. Proved in III. 4.

5. The same as 4, since the diagonals of a parm. bisect each other.

6. The diagonals AC, BD may be proved to pass through centre E. $\angle$EAB $=$ $\angle$EBA (I. 5.), $\angle$ECB $=$ $\angle$EBC (I. 15.) $\therefore$ in $\triangle$ABC one angle $=$ the sum of the other two $\therefore$ $\angle$ABC is a rt. $\angle$.

7. Since the chords are $\parallel$, a perp. to one is perp. to all $\therefore$ it bisects all, and is the locus of their mid. points.

8. Let AB, AC be the chds., DA the radius; DE, DF perps. to the chds. Then AE $=$ AF (I. 17.) $\therefore$ AB $=$ AC. The converse is proved by I. 7.

9. Since the join of the mid. points is perp. to the first chord, it passes through the centre; and since it joins the centre to the mid. point of the 2nd chord, it is perp. to that (III. 3.).

10. Let ABCD be the line, EF the perp. from the common centre. AF $=$ FD, and BF $=$ FC (III. 3.) $\therefore$ AB $=$ CD.

11. Half chord $= 12$ $\therefore$ distance $= \sqrt{15^2 - 12^2} = 9$ (II. 11.).

12. Let the perps. drawn from the centre E meet the chds. AB, CD in F, G. Draw EH $\parallel$ to the chds. $\angle$HEF $=$ alt. $\angle$EFB $=$ a rt. $\angle$. $\angle$HEG $=$ alt. $\angle$EGD $=$ a rt. $\angle$ $\therefore$ FEG is a str. line and is the join of the mid. pts.

13. Let CAB be the line, DF, EG perps. from centres. DE $=$ FG (II. 2.) $= \frac{1}{2}$CB (III. 3.).

14. The line of centres is perp. to AB (III. 2.) $\therefore$ it is perp. to CF (say at H) $\therefore$ CH $=$ HF and DH $=$ HE (III. 3.) $\therefore$ CD $=$ EF.

15. Let AEB, CED be the chds., F the centre. Let EF meet the circle in K, L. Let FG, FH be perp. to AB, CD. In $\triangle$s FAG, FCH, FG $=$ FH (I. 17.) $\therefore$ in $\triangle$s FEG, FEH, $\angle$FEG $=$ $\angle$FEH (I. 7.) $\therefore$ KL is one of the bisectors. But the bisectors of supplementary $\angle$s are at rt. $\angle$s $\therefore$ the other bisector is perp. to KL, and is therefore bisected by it (III. 3.).

16. Let AB, AD be the equal lines, FC, FE perps. from centre. In $\triangle$s ABF, ADF, $\angle$BAF $=$ $\angle$DAF (I. 7.) $\therefore$ FC $=$ FE (I. 16.).

17. In $\triangle$s ABD, ACE, $\angle$ABD $=$ $\angle$ACE (I. 5.), $\angle$ADB $=$ $\angle$AEC (I. 5.) and AB $=$ AC $\therefore$ BD $=$ EC (I. 16.).

18. Let AB be common chd., CED line of centres. CE $= \frac{1}{2}$CD $= \frac{1}{2}$radius (diagonals of rhombus bisect each other at rt. $\angle$s) $\therefore$ $AE^2 = r^2 - \frac{1}{4}r^2 = \frac{3}{4}r^2$ $\therefore$ $AB^2 = 3r^2$.

19. Let ABC, DEF be the parallels. GH, KL perps. to them through the centres. GK $= \frac{1}{2}$AC, and HL $= \frac{1}{2}$DF (III. 3.). But GK $=$ HL (II. 2.) $\therefore$ AC $=$ DF.

EXERCISES XXVIII.

1. Let D, E be the centres, then AED is a st. line. $\angle$ECA $=$ $\angle$EAC (I. 5.) $=$ $\angle$DBA (I. 5.) $\therefore$ DB is $\parallel$ to EC (I. 19.).

2. The join of centres DE passes through A. $\angle$ECA $=$ $\angle$EAC (I. 5.) $=$ $\angle$DAB (I. 3) $=$ $\angle$DBA (I. 5.) $\therefore$ DB is $\parallel$ to EC (I. 18.).

3. BC passes through the pt. of contact F. AB, AC produced go through the pts. of contact D, E. Perimeter of ABC $=$ AB $+$ BF $+$ AC $+$ CF $=$ AB $+$ BD $+$ AC $+$ CE $=$ 2AD $=$ a constant.

4. Let OP, OQ be tangents, C the centre. From $\triangle$s OPC, OQC by I. 17., OP $=$ OQ.

5. Let C be centre. Produce CP to S so that PS $=$ CP. The identically equal $\triangle$s CPO, SPO make up an equilateral $\triangle$. $\angle$POC $= \frac{1}{2}\angle$COS $= 30°$. Similarly $\angle$COQ $= 30°$ $\therefore$ $\angle$POQ $= 60°$. But the $\angle$s OPQ, OQP are equal $\therefore$ each of them is $60°$ $\therefore$ $\triangle$POQ is equilateral.

6. Let ADB touching at D meet the two radii in A, B. Let AE, BF be tangents, C the centre. By using I. 17. for the $\triangle$s FCB, DCB, we show that $\angle$FCD $= 2\angle$BCD. Similarly $\angle$ECD $= 2\angle$ACD $\therefore$ $\angle$FCD $+$ $\angle$DCE $= 2\angle$ACB $= 2$ rt. angles $\therefore$ FC $=$ CE are in a st. line. But the $\angle$s E, F are rt. $\angle$s. $\therefore$ AE is $\parallel$ to BF.

7. Take centre C. $\angle$CQA $=$ $\angle$CAQ (I. 5.) $=$ $\angle$QAP (hyp.) $\therefore$ by I. 18. AP is $\parallel$ to CQ, and consequently perp. to tangent at Q.

8. (1) Let A be the given point in which all touch the given st. line AB. Then DAC perp. to AB contains all the centres (III. 5., Cor. 2).

(2) If E be the given point in which a given circle, whose centre is F, is touched by a number of circles, the centres must all lie in FE (produced if necessary) III. 6.

EXERCISES XXIX.

1. Let AB, CD be the tangents, BE, ED radii to the pts. of contact. Let EF be parallel to AB and CD. ∠FEB = alt. ∠EBA = a rt. ∠ (I. 20.). Similarly ∠FED is a rt. ∠ ∴ BED is a straight line, *i.e.* the pts. of contact are the ends of a diameter.

2. The distance of each chord from the common centre is the radius of the inner circle ∴ the chords are equidistant from centre, and consequently equal (III. 10.).

3. Let BAC, DAE be the chords, F the centre, FG, FH perps. to BC, DE. From △s FAG, FAH, FG = FH (I. 16.) ∴ chord BC = chord DE (III. 10.).

4. Let PT be such a tangent, P the pt. of contact, C the centre. $CT^2 = CP^2 + PT^2$ = a constant + a constant ∴ CT is of constant length, and the locus of T is a circle with centre C.

5. Let TP, TQ be tangents; OP, OQ radii. ∠OPQ = ∠OQP (I. 5.) ∴ their complements are equal, *i.e.* ∠TPQ = ∠TQP.

6. In △s TOP, TOQ, ∠TOP = ∠TOQ (I. 17.).

7. Let AP, BQ be parallel tangents, touching at A, B; PQ a third line touching at R. Draw radii CA, CB, CR. CA, CB are perp. to parallel str. lines ∴ they are in a str. line. ∠PCR = $\frac{1}{2}$∠ACR (I. 17.). Similarly ∠QCR = $\frac{1}{2}$∠BCR ∴ ∠PCQ = $\frac{1}{2}$(∠ACR + ∠BCR) = a rt. angle.

8. Let AB, BC, CD, DA touch at E, F, G, H. As in xxviii. 4. tangents are equal ∴ AE + EB + DG + GC = AH + BF + DH + CF, *i.e.* AB + DC = AD + BC.

9. With the same figure as in 8, O being the centre, ∠AOE = ∠AOH (I. 7.). Similarly with the other angles ∴ ∠AOE + ∠EOB + ∠COG + ∠GOD = ∠AOH + ∠BOF + ∠FOC + ∠DOH, *i.e.* ∠AOB + ∠COD = ∠AOD + ∠BOC.

10. Let A be the centre of the outer, B of the inner circle. Let DBAC be the diameter of the inner circle which passes through A. Of all lines drawn from A to the inner circle AD is greatest, AC is least (III. 7. or 8.) ∴ the tangent drawn at D is the least chord and that at C is the greatest chord of the outer circle (III. 10.).

11. Let A be the given point, B the centre. Draw chords CAD, EAF each making an ∠ 45° with AB. Draw BG, BH perp. to the chords. These perps. are equal (I. 16.) ∴ CD = EF (III. 10.). Also ∠DAF is a rt. angle.

12. Let A, B be centres; D, E opposite ends of ∥ diameters DAF, EBG; C the pt. of contact. Join DC, EC. By I. 5. and I. 22. ∠ACD = $\frac{1}{2}$∠FAC and ∠ECB = $\frac{1}{2}$∠GBC. But ∠FAC = alt. ∠CBG (I. 20.) ∴ ∠ACD = ∠ECB = supplement of ∠ACE since ACB is a str. line ∴ DC and CE are in a str. line.

13. Let CT the common tangent meet AB in T. The tangents are equal ∴ ∠TCA = ∠TAC and ∠TCB = ∠TBC (I. 5.). In △ACB one angle = the sum of the other two ∴ ACB is a rt. angle.

EXERCISES XXX.

1. ∠CAE = ∠BDE (III. 12.). ∠ACE = ∠DBE (III. 12.). ∠AEC = ∠BED (I. 3.).

2. Let ABC be the triangle; P, Q, R pts. on the arcs BC, CA, AB respectively. The ∠s P, Q, R are the supplements of ∠s CAB, ABC, BCA (III. 13.) ∴ the sum of ∠s P, Q, R, CAB, ABC, BCA = 6 rt. angles ∴ sum of ∠s P, Q, R = 4 rt. angles.

3. Let the pentagon be ABCDE, P a point in arc AB. The ext. ∠ of a regular pentagon = $\frac{4}{5}$ rt. ∠ ∴ ∠EBA = $\frac{2}{5}$ rt. ∠ (I. 5. and I. 22.). ∠APE = ∠ABE (III. 12.) = $\frac{2}{5}$ rt. ∠. ∠EPD = ∠ECD (III. 12.) = $\frac{2}{5}$ rt. ∠. ∠APB = supplement of ∠AEB = $\frac{8}{5}$ rt. ∠ = the sum of the ∠s subtended by AE, ED, DC, CB.

4. In △ABC let D, E, F be the feet of perps., M, Q, R the mid. pts. of sides. In the rt. angled △BEC the mid. point of hypotenuse is equidistant from the vertices ∴ ∠MEB = ∠MBE (I. 5.). Similarly ∠REB = ∠RBE ∴ ∠REM = ∠RBM = ∠RQM (II. 2.) ∴ M, Q, R, E are concyclic (III. 13.). Similarly D, F lie on the circle MQR.

5. With the same figure as in the previous question, let N be the mid. point of OP, and NS parallel to OM, and therefore perp. to BC. $\triangle SMN = \triangle SON$ (II. 5.) $= \triangle SPN$ (II. 6.) $= \triangle SDN$ (II. 5.) $\therefore MS = SD$ (from the area of a $\triangle$). Thus N lies on the perpendicular bisector of the chord MD. Similarly N lies on the perpendicular bisector of the chord QE $\therefore$ N is the centre of the circle.

6. Let AB be the given base, C one position of the vertex, P another position. Let the circle through A, B, C cut AP in Q. Join BQ. $\angle AQB = \angle ACB$ (III. 13.) $= \angle APB$ (Hyp.). But this is impossible by I. 8.; P must lie on the arc ACB, *i.e.* the locus is the arc of segment which is on the given base, and which contains the given angle.

7. Let ABCD be a cyclic quadl., AB produced to E. $\angle CBE$ = supplement of $\angle CBA$ (I. 1.) $= \angle ADC$ (III. 13.).

8. $\angle EBC = 180° - \angle ABC = \angle ADC$. $\angle E$ is common $\therefore$ the $\triangle$s are equiangular to each other.

9. The angles at the point are together 360° $\therefore$ each is 120° $\therefore$ in order that there may be such a point each angle of the $\triangle$ must be less than 120° (I. 13.).

10. $\angle BAC + \angle BCA = 90° = \angle DAC + \angle DCA \therefore \angle BAC - \angle DAC = \angle DCA - \angle BCA$.

(2) Let BD, AC meet at E. $\angle BCA + \angle DAC = \angle BCA + \angle DBC$ (III. 12.) $= \angle BEA$ (I. 22.) $= 60°$. But $\angle BCA + \angle DCA = 90° \therefore$ by subtraction $\angle DCA - \angle DAC = 30°$.

11. Let A, B, C, D be the centres, E, F, G, H the pts. of contact, E lying on AB and so on. $\angle AEH = \angle AHE$ (I. 5.) $= \alpha$ say. $\angle BEF = \angle BFE = \beta$, $\angle CFG = \angle CGF = \gamma$, $\angle DGH = \angle DHG = \delta$. In $\triangle AEH$ $2\alpha = 180 - A$, and similarly for β, etc. But $A + B + C + D = 360°$ (I. 22., Cor.) $\therefore \alpha + \beta + \gamma + \delta = 180°$. Now $\alpha + \beta + HEF = 180°$, and $\gamma + \delta + HGF = 180°$ $\therefore$ by addition $\angle HEF + \angle HGF = 180°$, *i.e.* the figure EFGH is cyclic.

12. Let the quadl. be ABCD, the intersection of diagonals E. The $\angle$s subtended at centre by BC, AD are 2BDC, 2DCA (III. 11.) The sum of these $= 2\angle DEA$ (I. 22.) $= 180°$.

13. $\angle EFG = \frac{1}{2}B + \frac{1}{2}C$ (I. 22.), $\angle EHG = \frac{1}{2}A + \frac{1}{2}D$ (I. 22.) $\therefore \angle EFG + \angle EHG = \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D = 180°$ (I. 22., Cor.).

14. Suppose AB, DC meet in P; AD, BC in Q; and the circles meet in R. $\angle$ PRC = 180° − PBC (III. 13.) = $\angle$ CBA. Similarly $\angle$ QRC = $\angle$ CDA ∴ $\angle$ PRC + $\angle$ QRC = $\angle$ CBA + $\angle$ CDA = 180° (III. 13.) ∴ P, R, Q are in a str. line.

15. Complete the circle of which APB is a segment. Produce QB to meet the circle in R. $\angle$ Q = $\angle$ BPQ (I. 5.) = supplement of $\angle$ APB = $\angle$ ARB (III. 13.) ∴ if the figure were folded along AB, the arc ARB would pass through the point Q ∴ the locus of Q is an arc of an equal circle.

16. $\angle$ ADB = 90° = $\angle$ AEB ∴ the circle through B, D, A passes through E. $\angle$ DEC = supplement of $\angle$ DEA = $\angle$ ABD (III. 13.).

17. As in Question 16, a circle will go round ABDE ∴ $\angle$ ADE = $\angle$ ABE (III. 12.).

18. $\angle$ C + $\angle$ ABE = 180° (III. 13.), $\angle$ D + $\angle$ ABF = 180° (III. 13.) ∴ $\angle$ C + $\angle$ D + 180° = 360° (I. 1.) ∴ $\angle$ C + $\angle$ D = 180° ∴ CE is ‖ to DF (I. 19.).

19. The $\triangle$s ADE, BDE are equilateral ∴ $\angle$ ADB = 60° + 60° = 120°; $\angle$ BCA = $\frac{1}{2}$ADB (III. 11.) = 60°. Similarly $\angle$ BPA = 60° ∴ $\triangle$ BCP is equilateral.

20. $\angle$s in the same segt. of a circle are equal ∴ since the circles are equal, and AC = AD, $\angle$ in segt. CBA of circle CBA = $\angle$ in segt. DBA of circle DBA ∴ $\angle$ ABC = $\angle$ ABD ∴ BDC is a str. line.

21. Let C be the centre, 2$\angle$C = $\angle$ O = a constant, and each of the sides CP, CQ = the constant radius ∴ the base PQ is of constant length.

22. Let ABCD be the quadl., CD produced to E, F the intersection of the bisectors. $\angle$ EDF = $\frac{1}{2}\angle$ EDA = $\frac{1}{2}$ int. oppte. $\angle$ ABC = $\angle$ CBF ∴ $\angle$ CBF + $\angle$ CDF = $\angle$ EDF + $\angle$ CDF = 180° ∴ F lies on the circle BCD.

23. Take any point F on the circumference not in the arc BDC. $\angle$ EDB = 180° − $\angle$ BDC = $\angle$ BFC = $\frac{1}{2}\angle$ BAC (III. 11.).

24. Let CB meet DE in H. The $\angle$s at G, E being rt. $\angle$s, the quadl. GFED is cyclic. Similarly for CFBD ∴ $\angle$ DEG = $\angle$ DFG (III. 12.) = 90° − $\angle$GDF = $\angle$FDB = $\angle$FCB (III. 12.) = $\angle$EHB (I. 20.), (since FC is ‖ to ED) ∴ EG is ‖ to BC.

25. $\angle$ DRC = 180° − $\angle$ P − $\angle$ Q (I. 22.) = 180° − $\angle$ CAP − $\angle$ DAQ (I. 5.) = $\angle$ CAD = $\angle$ CBD (I. 7.) ∴ D, B, R, C are concyclic.

26. Let ABCD be a cyclic quadl. Let a str. line meet AB, DC, AD, BC in F, G, H, K respectively. By hypothesis ∠ DHK = ∠ CKH. But ∠ DHK = ∠ HGD + ∠ GDH (I. 22.) = ∠ HGD + ∠ B. Also ∠ CKH = ∠ BFG + ∠ B (I. 22.) ∴ ∠ BFG = ∠ HGD.

27. In the quadl. ABDC, ∠ A is constant (AB, AC being given tangents). Also ∠ BDC is constant (III. 12.) ∴ the sum of the other two ∠s is constant (I. 22. Cor.).

EXERCISES XXXI.

1. Let AB, DC be parallel. Then ∠ BAC = alt. ∠ ACD (I. 20.) ∴ arc BC = arc AD (III. 14.) ∴ chord BC = chord AD (III. 15.). Also the whole arc ADC = whole arc DCB ∴ chord AC = chord DB (III. 15.).

2. Let tangent at C be ∥ to AB. Let CDE be the radius cutting AB at D. CE is perp. to the tangent and therefore to AB ∴ D is the mid. point of AB. From I. 4. AC = CB ∴ arc AC = arc CB (III. 16.).

3. Let AEB, CED be any chords containing a constant ∠. The sum of the angles at B and C = ∠ E = a constant ∴ the sum of the arcs AC, BD is constant.

4. With the same figure the ∠ E = ∠ B + ∠ C at circumference = an ∠ at the circumference standing on the sum of arcs AC, DB = an ∠ at the centre on half the sum of the arcs.

5. Let AB, CD intersect at an external pt. E. Join BC. ∠ E = ∠ ABC − ∠ BCD (I. 22.) = an angle at circumference standing on an arc AC − BD = an angle at centre standing on half that arc.

6. ∠ ECD = ∠ EAB (III. 15.) = alt. ∠ EFD (I. 20.) ∴ CE = EF (I. 6.).

7. Arc BD = arc DC, since the ∠s at A are equal; ∴ chord BD = chord DC (III. 15.) = DE (*Hyp.*) ∴ ∠ DBE = ∠ DEB (I. 5.) = $\frac{1}{2}$∠A + ∠ABE (I. 22.) = ∠DAC + ∠ABE = ∠DBC + ∠ABE (III. 12.) ∴ ∠ CBE = ∠ ABE.

8. ∠ C is constant (III. 12.), since A and B are fixed points. Similarly ∠ D is constant. The 3rd angle of the △CBD must be constant.

9. The ∠s APQ, AQP are constant (III. 12.) ∴ by I. 22. ∠ RAQ is constant ∴ the arc RQ, and consequently the chord RQ, is constant (III. 14., 15.).

10. In $\triangle$s CED, CAB $\angle$E = $\angle$EAC = $\angle$CAB (Hyp.); $\angle$CDE = 180° − $\angle$CDA = $\angle$ABC. Also CE = CA ∴ DE = AB (I. 16.).

11. AB = CD ∴ arc AB = arc CD (III. 16.) ∴ $\angle$ACB = $\angle$DAC (III. 15.) ∴ AD is ∥ to BC (I. 18.). Also (by addition of arc AD) the arc DAB = arc CDA ∴ chord DB = chord CA (III. 15.).

EXERCISES XXXII.

NOTE.—*It follows from III.* 17. *that a chord which subtends a right angle at the circumference is a diameter.*

1. On the hypotenuse AB of $\triangle$ABC let a circle be described. It must pass through C. For if it cut AC at D, the $\angle$ADB would be a rt. angle (III. 17.) and so equal to $\angle$ACB, which is impossible (I. 8.).

2. As in Question 1, C must lie on the circle whose diameter is AB.

3. Join CB. Arc AC + arc BD = arc subtended by the sum of the $\angle$s C and B at the circumference = arc subtended by a rt. $\angle$ (I. 22.) = a semicircle. It might also be proved by drawing AF ∥ to CD, and proving arc FD = arc AC (I. 20. and III. 14.).

4. OQP is a rt. $\angle$ (III. 17.) ∴ PQ is a tangent (III. 5.).

5. CBA, ABD are rt. $\angle$s (III. 17.) ∴ C, B, D are in a str. line (I. 2.).

6. Draw CF perp. to tangent at A, and CG ∥ to FA. Join AD. $\triangle$s BGC, BDA are identically equal (I. 16.) ∴ BG = BD. CD = CB − BD = AB − BG = AG = CF (II. 2.).

7. Since the arcs are equal, the chords are equal (III. 15.) ∴ they are equidistant from the centre (III. 10.) ∴ they touch a concentric circle.

8. Let BC be the diameter, A the point of contact. $\angle$PAQ = $\angle$BAC = 90° (III. 17.) ∴ PQ is a diameter.

9. Chord AB = chord AE (radii) ∴ arc AB = arc AE (III. 16.) arc AD = arcs AE and ED = arc EC ∴ chord AD = chord EC (III. 15.). Also since arc AE = arc DC, $\angle$EDA = alt. $\angle$DAC (III. 15.) ∴ ED is ∥ to AC.

10. The mid. pt. of the hypotenuse of a right-angled $\triangle$ is equidistant from the vertices ∴ QD = QC ∴ $\angle$QDC = $\angle$C (I. 5.)

$= \angle$ QRP (II. 2.) $\therefore$ $\angle$ QRP + $\angle$ QDP = $\angle$ QDC + $\angle$ QDP = 2 rt. $\angle$s $\therefore$ QRPD is cyclic.

11. In $\triangle$AFB, G is the intersection of the perps. AE, BD $\therefore$ G is the orthocentre $\therefore$ FG is perp. to AB.

Or, let FG meet AB at H. In the cyclic quadl. DGFE, $\angle$DEG = $\angle$DFG (III. 12.). And in the cyclic quadl. ADEB, ext. $\angle$DEF = int. oppte. $\angle$DAB. But $\angle$BHF = $\angle$DFG + $\angle$DAH (I. 22.) $\therefore$ $\angle$BHF = $\angle$DEG + $\angle$DEF = a rt. $\angle$.

12. Let C be the centre of the circle Q, CD a diameter of the circle P. Let A, B be the pts. of intersection of the circle. $\angle$s CAD, CBD are right $\angle$s (III. 17.) $\therefore$ AD, BD are the tangents at A, B (III. 5.) *i.e.* the tangents at A, B meet at D.

13. Let AC be the diameter of smaller, AB of larger circle: AQP the chord. The $\angle$ at Q is a rt. $\angle$ (III. 17.) $\therefore$ Q is the mid. pt. of AP (III. 3.).

14. Let DOC be perp. to the fixed diameter AB. $\angle$OPA = $\angle$OAP = 90° − $\angle$APN = $\angle$BPN $\therefore$ the bisector of $\angle$OPN is the bisector of $\angle$APB $\therefore$ it passes through C the mid. pt. of the arc AB (III. 14.). If P were on the other side of AB, the bisector would pass through D.

EXERCISES XXXIII.

1. Let ACB be the arc, C the mid. point, AE the tangent, CE perp. to AE. CA = CB (III. 15.) $\therefore$ $\angle$CAB = $\angle$CBA = $\angle$CAE (III. 18.) $\therefore$ CD = CE (I. 16.).

2. Let ABCD be the circle, A the point, BC the chord, AD the diameter. Join BD. $\angle$ABD = 90° (III. 17.) $\therefore$ complement of $\angle$ABC = $\angle$CBD = $\angle$CAD (III. 12.) Also $\angle$ACB = $\angle$ADB (III. 12.) = complement of $\angle$BAD.

3. $\angle$ATP + 2$\angle$BPT = $\angle$ATP + $\angle$BPT + $\angle$PAB (III. 18.) = $\angle$ABP + $\angle$PAB (I. 22.) = 90° (III. 17.).

4. $\angle$BAD = $\angle$C (III. 18.) = 60°. Similarly $\angle$ABD = 60° $\therefore$ $\triangle$ABD is equilateral.

5. $\angle$APC = $\angle$CAP (I. 5.) = $\frac{1}{2}$ $\angle$PCD (I. 22.) = 30°. Similarly $\angle$B = 30° $\therefore$ $\angle$APC = $\angle$B $\therefore$ AP touches the circle BCP (III. 18.).

6. Draw AD perp. to BC. The semicircle on AB passes thro. D (III. 17.). Similarly the semicircle on AC passes thro. D.

7. Draw a str. line FAG touching the circle ABC at A. ∠DAG = ∠FAB (I. 3.) = ∠ACB (III. 18.) = ∠AED (I. 20.) ∴ circle EAD touches FG at A ∴ the circles touch each other at A.

8. Let the circles ABC, AED touch at A, FAG being the common tangent. Let BAD, CAE be the lines. ∠CBA = ∠CAG (III. 18.) = ∠FAE (I. 3.) = ∠ADE (III. 18.) ∴ BC is ‖ to ED (I. 18.).

9. With internal contact, ∠CBA = ∠CAG (III. 18.) = ∠ADE (III. 18.) ∴ BC is ‖ to ED (I. 19.).

10. ∠TPS = ∠SPR (III. 15.) = ∠PRQ (I. 5.) ∴ PT is a tangent (III. 18.).

11. Draw str. line FAG to touch the circle ABC at A. ∠FAB = ∠ACB (III. 18.) = ∠AED (I. 20.) ∴ circle ADE touches FG at A (III. 18.), and so touches the circle ABC.

12. The circles touch at A, EAF being the common tangent, BC the chord touching at D. AC cuts the inner circle at H. ∠BAD = ∠EAD − ∠EAB = ∠AHD − ∠C (III. 18.) = ∠HDC (I. 22.) = ∠DAH (III. 18.).

EXERCISES XXXIV.

1. Draw two circles with the given radius and with the given points for centres. The intersection gives the centre of the required circle.

2. Draw two perp. diameters and join their ends. I. 4. proves the sides equal. III. 17. proves the ∠s right ∠s.

3. Draw two perp. diameters. Draw the bisectors of the angles between them. The 8 arcs thus obtained are equal (III. 14.). Hence the 8 chords are equal (III. 15.). III. 15. will prove the angles of the octagon all equal.

4. Draw the chord at rt. ∠s to the join of the given point to the centre. Prove by III. 3.

5. With centre A on the circle, and with radius equal to the given length, describe a circle cutting the given circle in B. From C the centre of the given circle draw CD perp. to AB. With centre C and radius CD describe a circle, and from the given point O draw a tangent to this circle. This tangent can be proved to be the required line (III. 10.).

6. Let O be the point, C the centre. Describe a circle with centre C and radius equal to the given distance. A tangent from O to this circle is the required line.

7. Draw a diameter perp. to the given line: the tangents at the ends of this diameter are || to the given line.

8. Draw a diameter || to the given line: the tangents at the ends of this diameter are perp. to the given line.

9. Join the centre C to the given pt. A. Draw a chord BAD perp. to CA. Draw CG perp. to any other chord EAF. CGA is a rt. $\angle$ $\therefore$ CAG $<$ a rt. $\angle$ (I. 9.) $\therefore$ CA $>$ CG (I. 11.) $\therefore$ BD $<$ EF (III. 11.).

10. Let O be the given point, A the centre. Draw the diameter BC perp. to OA. The circle with radius OB is the one required (I. 4.).

11. Let A be the given point, CBD the given line, B the point of contact. BE drawn perp. to CD must contain the reqd. centre. Make $\angle$BAF equal to $\angle$ABE, and let AF meet BE in F. AF = BF (I. 6.) $\therefore$ the circle with radius FA or FB is the one required.

12. The centre is at the intersection of the diagonals, and the radius is half of either diagonal.

13. From the centre A draw AB perp. to the given str. line BC. Cut off BC equal to half the given length. Through C draw CDE || to BA to meet the circle in D and E. Draw a chord DG || to CB, cutting AB at F. DG = 2DF (III. 3.) = 2CB (II. 2.) = the given length. Another chord could be similarly drawn from E.

14. Let AB, CD be ||, and AC the third line. Bisect the angles A and C by AE, CE. Draw EF, EG, EH perp. to AC, CD, AB. EF = EH in $\triangle$s AFE, AHE (I. 16.). Similarly EF = EG $\therefore$ E is the required centre.

15. Draw str. lines || to the given str. lines at a distance from them equal to the given radius. The intersection of these gives the reqd. centre. There are 4 solutions.

16. From the given centre A draw AB perp. to the given str. line BC. Cut off BC equal to half the given length. Then AC is the radius of the reqd. circle.

17. Let ABC be the circle, O the centre. With centre A and radius AO describe a circle cutting the given circle in D, E. Produce AO to meet the circle in F. DEF is the reqd. $\triangle$. The $\triangle$AOD is equilateral, $\therefore \angle$DOA $= 60°$. Thus each of the $\angle$s DOE, EOF, FOD is $120°$ $\therefore$ $\triangle$DEF is equilateral (III. 14. and 15.).

18. Draw 6 radii, so that each of the 6 angles at the centre is $60°$ (as in Question 17.). The sides of the hexagon are equal (III. 14. and 15.). The angles are proved equal as in Ex. L., 3.

19. Bisect each of the angles at the centre in Question 18.

20. On the given base describe a segment containing the given angle (III. 23.). Find where the arc is cut by a line drawn $\parallel$ to the base at the distance of the given altitude.

21. Let ABC be the goal line, AB the goal, CDE the line in which the ball is taken out. On AB describe a segment of a circle ABDE containing the given $\angle$. D or E is the reqd. point.

(2) Bisect AB at F, and draw FG $\parallel$ to CD. With centre B and radius FC, cut FG at G. The circle with centre G and radius FC will touch CD at some point H; and AHB may be proved to be the maximum subtended $\angle$ (I. 8.).

22. See Question 5.

23. Make $\angle$BAC equal to the given vertical $\angle$. Make AB, AC each equal to half the given perimeter. Make ABE, ACE rt. $\angle$s. Describe a circle BDC with centre E. Describe a circle with centre A and radius the given altitude. Draw DF to touch both these circles (Exercises xxxvi. 1.), and to meet AB, AC at H, K. The perimeter of the $\triangle$AHK $=$ AH $+$ HD $+$ AK $+$ KD $=$ AH $+$ HB $+$ AK $+$ KC (tangents equal) $=$ AB $+$ AC $=$ given perimeter. Also AF $=$ given altitude: and A $=$ given vertical $\angle$.

24. Let AB be the given side, BAC the given $\angle$. With centre A and the given altitude for radius, describe a circle; and from B draw BDC touching it at D. ABC is the reqd. $\triangle$.

25. Bisect AB the common chord by DCE perp. to it. With centre A and radius equal to each of the given radii in turn, cut DCE at D, E. These are the required centres.

26. Let A, B be the centres, $r_1 r_2$ the radii of the given circles, r the radius of the reqd. circle. With centre A and radius $r_1 + r$, describe a circle. With centre B and radius $r_2 + r$, describe a circle. Let C be a pt. of intersection. The circle whose centre is C and radius r is the one required. Another solution may be obtained by using $r - r_1$, $r - r_2$ for AC, BC provided r is large enough. In the 1st case AB must be less than $2r + r_1 + r_2$, in the 2nd case AB must be less than $2r - r_1 - r_2$.

27. Let A be the first given point, B the given point of contact on the circle with centre C. Let DE which bisects AB at rt. ∠s meet CB at E. The reqd. centre is E, radius EA. A may be internal or external.

28. Let A be the first given point, B the given pt. in the line BC. Draw DE bisecting AB at rt. ∠s, and let it meet in E the perp. to BC drawn from B. The required centre is E.

29. Let A be a common point; B, C centres. Bisect BC at D. Draw EAF perp. to DA. Draw CG, BH perp. to EF; CM, DK perp. to DA, BH. In △s BKD, DMC, KD = MC (I. 16.) ∴ HA = AG (II. 2.) ∴ 2HA = 2AG, *i.e.* FA = AE.

30. Let A be the pt. where the given str. line is met by the bisector of the ∠ between the two given lines BEC, CFD. Draw AB, AD perp. to BC, CD; and mark off BE, DF each equal half the given length. AE or AF is the reqd. radius.

31. Any circle described with its centre at the incentre of the △ has this property, since the chords cut off are equidistant from the centre, and therefore equal.

32. In the second circle place a chord of the reqd. length. Draw a perp. to this chord from the centre. With the perp. for radius describe a concentric circle. Draw (Ex. xxxvi. 1.) a common tangent to this and the first circle.

33. In the given circles place chords of the given lengths. Draw perps. from the centres. Describe circles with these perps. as radii. Draw a common tangent to these two circles. [Impossible when the given chords are greater than the corresponding diameters of the given circles.]

34. Cut off a segment containing an angle of 60°. The other segment contains an angle of 120°.

35. Cut off a segment containing an angle of 30°.

36. Let A, B be the given points. On AB as diameter describe a circle cutting the given str. line in C, D.

37. Let A be the given pt., BC the given chord, O the centre. On OA describe a semicircle cutting BC at D. Then ADE is the required chord. OD is perp. to AE (III. 17.) ∴ AD = DE (III. 3.).

38. Let A, B be the centres. Draw any radii AC, BD. Draw CE, DF perp. to these, and equal to the given lengths. Join AE, BF. Describe circles with centres A, B and radii AE, BF, and let them intersect at G. The tangents GH, GK are of the required lengths. For GH = EC, and GK = FD (I. 17.).

39. Draw two radii AB, AC including an angle supplementary to the given ∠. Let the tangents at B, C meet at D. The point D is the one required. For B, C being rt. ∠s the angle CDB is the supplement of ∠CAB.

(2) ∠DAB = $\frac{1}{2}$∠CAB = a constant; and AB is of constant length ∴ AD is constant and the locus of D is a concentric circle.

40. Draw radii AB, AC, including a rt. ∠. BC is the required chord: for ∠ABC = 45° ∴ ∠BAD = 45° ∴ AD = DB = $\frac{1}{2}$BC, where AD is perp. to BC.

41. Let A, B be the given points. On AB describe a segment containing the given ∠. The pt. or pts. in which this meets the given circle will give the required pt.

42. Let A be the given pt. Take any pt. B on the outer circle. With centre B and radius the given length, cut the inner circle at C. Produce BC to meet the circles in D, E. From the common centre O draw a perp. OF to BE. With centre O and radius OF describe a circle. Draw a tangent from A to this circle. This tangent is the line required (III. 10.).

43. Let A be the point. Cut off a segment BDC containing an ∠ equal to the given ∠ (III. 24.). From the centre E draw to BC a perp. EF. Describe a concentric circle with radius EF. Draw AGHD a tangent to this circle at H. GD = BC, since EH = EF (III. 10.) ∴ arc GD = arc BC (III. 16.) ∴ ∠GCD = ∠BDC (III. 15.).

44. Through A, one pt. of intersection, draw any such line CD. From the centres E, F draw perps. EG, FH. Draw EK $\parallel$ to CD. CD $=$ 2GH (III. 3.). Also GH $=$ EK $<$ EF (I. 11.) $\therefore$ CD $<$ 2EF unless CD is $\parallel$ to EF $\therefore$ the maximum position is $\parallel$ to EF.

45. Draw BC perp. to AB to meet the circle in C. The circle on AC as diameter touches the given circle (III. 6.). Draw any line AEF to meet the circles. $\angle$ACB $=$ $\angle$AEB (III. 12.) $>$ $\angle$AFB (I. 8.) $\therefore$ C is the point required.

46. Let A be the centre of the given circle, BC the given str. line, C the given pt. In BC produced make CD equal to the radius of the given circle. Draw EB bisecting AD at rt. $\angle$s. AB $=$ BD (I. 4.), and AF $=$ CD (cons.) $\therefore$ BF $=$ BC, and the circle described with centre B is the one required.

EXERCISES XXXV.

1. Let ABC be an equilateral $\triangle$, I the incentre. In $\triangle$s IFB, IDC, $\angle$IBF $= 30° =$ $\angle$ICD, $\angle$F $= 90° =$ $\angle$D, and IF $=$ ID (radii) $\therefore$ IB $=$ IC $=$ similarly IA.

2. Let DEF be a circumscribing equilat. $\triangle$, ABC an inscribed equilat. $\triangle$, DE touching the circle at C, EF at A, FD at B. $\angle$FBA $=$ $\angle$FAB $=$ $\angle$ACB (III. 18.) $= 60°$ $\therefore$ $\triangle$FAB is equilat. $\therefore$ FA $=$ AB. Similarly AE $=$ AC $=$ AB $\therefore$ FE $=$ 2AB.

3. Let $\angle$ACB $=$ $\angle$DFE $\therefore$ arc AB $=$ arc DE $\therefore$ chord AB $=$ chord DE (III. 15.) $\therefore$ the $\triangle$s are equal in all respects (I. 16.).

4. AB, AC are produced to D, E. Bisect $\angle$s CBD, BCE by BF, CF. From F draw perps. to the sides. These are equal (I. 16.) $\therefore$ the circle with any of these three for radius is the one required.

5. PC is perp. to AC (III. 17.), and BT is perp. to AC (hyp.) $\therefore$ PC is $\parallel$ to BT. Similarly PB is $\parallel$ to CT.

6. EB, BD are the bisectors of supplementary $\angle$s. $\angle$EBD $= \frac{1}{2}$ of 2 rt. $\angle$s $=$ a rt. $\angle$. Similarly $\angle$ECD is a rt. $\angle$ $\therefore$ ED is the diameter of a circle through B and C.

7. Let I be the incentre of $\triangle$ABC which has a rt. $\angle$ at A. ID, IE, IF perp. to BC, CA, AB. Tangents are equal $\therefore$ BF $=$ BD, and CE $=$ CD, also FA $=$ IE (II. 2.) $= r$, and EA $=$ IF $= r$ $\therefore$ BF $+$ FA $+$ CE $+$ EA $=$ BC $+ 2r$, *i.e.* AB $+$ AC $=$ the sum of the diameters.

8. In the cyclic quadl. ODCE $\angle ODE = \angle OCE$ (III. 12.) $= 90° - \angle BAC$. Similarly $\angle ODF = \angle OBF = 90° - \angle BAC$ ∴ DO bisects $\angle EDF$. Similarly for EO ∴ O is the incentre of $\triangle DEF$.

9. Draw DE, DF, DG perp. to BC, CA, AB. DE = DG (I. 16.). Similarly DE = DF ∴ DG = DF ∴ in $\triangle$s DAG, DAF, $\angle DAG = \angle DAF$ (I. 17.) ∴ DA bisects $\angle BAC$ and consequently contains the incentre.

10. Let ABC be the equilat. $\triangle$. $\angle BOA = 2\angle C$ (III. 11.) $= 120°$ ∴ $\angle BOD = 60°$. $\angle ADB = \angle ACB$ (III. 12.) $= 60°$ ∴ $\triangle BOD$ is equilateral.

EXERCISES XXXVI.

4. Proved in Ex. xxxv. 8.

5. Let ABC be the $\triangle$, AGD, BGE, CGF the medians. Let BC be trisected at H, K. Let L, N be the mid. pts. of AG, BH. $\triangle ABN = \triangle ANH = \triangle AHD$ (II. 6.) ∴ $\triangle ABH = \frac{2}{3}\triangle ABD$. Similarly $\triangle AGB = \frac{2}{3}\triangle ABD$ ∴ $\triangle ABH = \triangle AGB$ ∴ GH is ∥ to AB (II. 7.). Similarly GK is ∥ to AC ∴ $\angle HGK = \angle BAC$ = a constant ∴ the locus of G is the arc of a segment on HK.

6. Let A be the fixed pt., BC the given str. line. Draw AB perp. to BC and produce AB to D, making BD equal to AB.

Let O be the centre of one of the circles. Then OD = OA (I. 4.) ∴ D lies on the circle. Similarly D lies on each of the circles.

8. Let the circle touch BC at D. AF = AE (tangents), and similarly BF = BD, CE = CD ∴ AF + AE = AB + BD + CD + AC = $2s$ ∴ s = AF = AE.

9. BD = BF = AF − AB = $s - c$; similarly CD = $s - b$.

10. Proved in Ex. xxxv. 5.

11. L is the mid. pt. of BC (III. 3.). But the mid. pt. of one diagonal of a parm. is the mid. pt. of the other ∴ L is the mid. pt. of PT ∴ OL = $\frac{1}{2}$AT (Ex. xx. 1.).

12. Let the altitudes be AD, BE, CF; the orthocentre T. $\angle BTD = 90° - \angle TBD = \angle BCE$ (from $\triangle BCE$). Similarly $\angle DTC = \angle ABD$ ∴ $\angle BTC = \angle C + \angle B = 180° - A$ = a constant ∴ the locus is the arc of a segment on BC.

EXERCISES XXXVII.

1. The str. line perp. to the given str. line at the given pt. contains all the centres (III. 5. Cor. 2.).

2. Join the mid. pts. to the centre. These joins are perp. to the chords (III. 3) and are therefore equal (III. 10.) $\therefore$ the locus is a concentric circle.

3. Join the fixed pt. to the common centre. This line subtends a rt. $\angle$ at a pt. of contact $\therefore$ the locus of the pts. of contact is a circle whose diameter is this line (III. 17.).

4. Let A be the fixed pt., C the centre of the circle, CP perp. to a chord. P is the mid. pt. of the chord, and the locus of P is the circle whose diameter is AC (III. 17.).

5. Produce BC to E making $CE = BC$. A, C are the mid. pts. of BD, BE $\therefore$ AC is $\parallel$ to DE $\therefore$ $\angle BDE = \angle BAC =$ a constant. $BE = 2BC =$ a constant $\therefore$ the locus of D is the arc of a segment of a circle on BE.

6. Let A, B be the fixed pts. P the pt. of contact of the circles. Draw the common tangent at P meeting AB at T. Tangent $TA = TP = TB$ $\therefore$ P lies on the circle whose diameter is AB.

7. Let E be the intersection of AD, BC. $\angle AEB = \angle CAD + \angle ACB = 90° +$ a constant $\angle$ (III. 17. 12.) $\therefore$ the locus of E is the arc of a segment on AB.

8. Let BC be the given base, A the vertex, O the circumcentre. $\angle BOC = 2\angle A =$ a constant $\therefore$ the locus of O is the arc of a segment on BC.

9. Let I be the incentre, BC the given base, A the vertex. Let AI be produced to D. $\angle BIC = \angle BID + \angle DIC = \frac{A}{2} + \frac{B}{2} + \frac{A}{2} + \frac{C}{2}$ (I. 22) $= \frac{A}{2} + 90° =$ a constant $\therefore$ the locus of I is the arc of a segment on BC.

10. Let BC be the base, K the excentre. $\angle BKC = 180° - \angle KBC - \angle KCB = 180° - \left(90° - \frac{B}{2}\right) - \left(90° - \frac{C}{2}\right) = \frac{B+C}{2} = 90° - \frac{A}{2}$ $=$ a constant $\therefore$ the locus of K is the arc of a segment on BC. (It is the remaining arc of the circle mentioned in the preceding.)

11. Let the diagonals intersect at E. Bisect AB at F. E is the mid. pt. of AC, F of AB $\therefore$ EF is $\parallel$ to CB $\therefore$ $\angle$AEF $=$ $\angle$ACB $=$ a constant. Also AF $= \frac{1}{2}$AB $=$ a constant $\therefore$ the locus of E is the arc of a segment on AF.

12. Let AB be the edge of the ruler sliding on CA, CB; D its mid. pt. CD $= \frac{1}{2}$AB (Ex. xviii. 9. or III. 17.) $=$ a constant $\therefore$ the locus of D is a circle with centre C.

13. Let the bisectors meet at E. $\angle$EAC $+ \angle$ECA $= \frac{1}{2}(\angle$BAC $+$ $\angle$ACD$) = 90°$ (I. 20.) $\therefore$ $\angle$AEC $= 90°$ $\therefore$ E lies on the circle whose diameter is AC.

14. Let R be the pt. of intersection. $\angle$P $=$ $\angle$PAC, $\angle$Q $=$ $\angle$QAD (I. 5.). $\angle$CRD $= 180° - \angle$P $- \angle$Q (I. 22) $= 180° - \angle$PAC $-$ $\angle$QAD $= \angle$CAD $= \angle$CBD $\therefore$ R lies on a circle through C, B, D.

15. Let APB be the segment on AB, Q the centre of the circle of which this is a segment. Let APB be on the side of AB remote from C the centre of the given circle. AB is the common chord of the two circles $\therefore$ CQ bisects AB at rt. $\angle$s $\therefore$ $\angle$CQA $= \frac{1}{2}\angle$AQB $= \angle$APB (III. 11.) $=$ a constant $\therefore$ Q is on a circle through A and C. By drawing AB in different positions it is found that two circles are obtained.

EXERCISES XXXVIII.

1. $r = \sqrt{26^2 - 24^2} = 10.$

2. Tangent $= \sqrt{37^2 - 35^2} = \sqrt{72 \times 2} = 12.$

3. The two radii and the chord form an equilat. $\triangle$ $\therefore$ $r = 6.$

4. Distance $= \sqrt{65^2 - 63^2} = 16.$

5. Distance of 1st chord fr. centre $= \sqrt{85^2 - 36^2} = 77.$ Distance of 2nd chord fr. centre $= \sqrt{85^2 - 51^2} = 68$ $\therefore$ distance between chords $= 77 \pm 68 = 145$ or 9

6. Chord $= 2\sqrt{13^2 - 5^2} = 24.$

7. Chord $= 2\sqrt{27^2 - 12^2} = 6\sqrt{65} = 48{\cdot}37$ decimetres; chord of half arc $= \sqrt{9 \times 65 + 15^2} = 9\sqrt{10} = 28{\cdot}46$ decimetres.

8. $32^2 + (r - 8)^2 = r^2$ $\therefore$ $16r = 8^2 + 32^2$ $\therefore$ $r = 68.$

9. Triangle formed by centres has base 20, sides $10 + r$, $10 + r$, and altitude $20 - r$ $\therefore$ $(20 - r)^2 + 10^2 = (10 + r)^2$ $\therefore$ $400 - 40r = 20r$ $\therefore$ $r = \frac{20}{3} = 6\frac{2}{3}.$

10. Chord $= 2\sqrt{25^2 - 24^2} = 14$.

11. $r = 4$.

12. Chord $= 4 \cdot 8$.

13. Distance $= \sqrt{35^2 - 28^2} = 21$.

14. Distance $= \sqrt{70^2 - 24^2} = \sqrt{4324} = 65 \cdot 76$.

15. The mid. point of chord is the centre of circle. Since 3 equal str. lines are drawn fr. it to circumf. $\therefore r = 5$.

16. Distance $= \sqrt{2 \cdot 6^2 - 2 \cdot 4^2} = 1$.

17. Distance of 1st chord fr. centre $= \sqrt{5^2 - 3^2} = 4$. Distance of 1st chord fr. centre $= \sqrt{5^2 - 4^2} = 3 \; \therefore$ distance apart $= 4 \pm 3 = 7$ or 1.

18. 3·57 each.

19. Distance $= 2 \cdot 1$.

20. Distance $= 5 \cdot 74$.

21. 13·86 cms. or 5·2 inches.

22. Let O be the centre, A a pt. on the circumference. Cut the circumference at B, C by a circle with centre A and radius AO. Let AO, BO, CO meet the circle in D, E, F. AOB, AOC are equilat. $\triangle$s by construction $\therefore \angle BOC = 120^\circ \; \therefore \angle COE = 60^\circ \; \therefore$ by I. 3. all the angles at O are equal $\therefore$ the 6 arcs are all equal. The $\angle$s of the hexagon are equal; for each stands on two-thirds of the circumference.

23. 3·75.

24. $r = \dfrac{ab}{\sqrt{4a^2 - b^2}}$. [In rt.-angled $\triangle$ formed by radius, tangent, and line joining the external pt. to the centre $\dfrac{b^2}{4} =$ product of segments of hypotenuse $= \sqrt{r^2 - \dfrac{b^2}{4}}\sqrt{a^2 - \dfrac{b^2}{4}} \; \therefore$ by squaring we get r.

25. 6·3.

26. QC $= b - c$. Let OA $= x$. Then OD $= x - a$. OQ $=$ OB $-$ QB $=$ OB $-$ QC $= x - b + c \; \therefore$ by II. 11. $(x - b + c)^2 - (x - a)^2 = c^2 \; \therefore \; (c + a - b)(2x - a - b + c) = c^2 \; \therefore \; 2x(c + a - b) + c^2 - 2bc + b^2 - a^2 = c^2 \; \therefore \; x = \dfrac{a^2 + 2bc - b^2}{2(c + a - b)}$. When $b = 5$, and $a = c = 3$, OA $= 7$, and QC $= 2$.

27. Let A, B be the posts, T the tree. Sets of sufficient measurements, (1) lengths of AT and BT, (2) AT and $\angle$BAT, (3) BT and $\angle$ABT, (4) $\angle$s ABT and BAT.

28. $60° + \angle BCA < 180°$ (I. 9.) $\therefore \angle BCA < 120°$. The least distance of C from AB is the perpendicular, *i.e.* $AC = 4\sqrt{3} = 6{\cdot}93$ cms.

29. Each $\angle = 34\frac{1}{2}°$ approximately. Prove III. 12.

30. Let $PA = a$, $PB = b$, $PQ = x$. $x^2 + a^2 + x^2 + b^2 = AQ^2 + BQ^2 = AB^2 = (a+b)^2 \therefore x^2 = ab$. Thus rect. $PA \,.\, PB = PQ^2 \therefore$ the maximum is when Q is as far as possible from P, *i.e.* when $PA = PB$.

(2) The semiperimeter is given, viz. 1000 yds. By the property just proved the rectangle is a maximum when the adjacent sides are equal $\therefore$ each side $= 500$ yds.

31. Make $PA = 11$, $PB = 7$, and let APB be a str. line. Draw a semicircle on AB, and draw PQ perp. to AB to meet the circumference in Q. $QP^2 = PA \,.\, PB = 77$. $QP = \sqrt{77} = 8{\cdot}78$.

32. Describe an equilateral $\triangle$AOB. $\angle AOB = 60°$. Describe a circle with centre O and radius OA. The larger segment is the one required. For the $\angle$ in it $= \frac{1}{2} \angle AOB$ (III. 11.) $= 30°$.

33. 180° in each case.

34. Let P be the external point, PT a tangent. $TP^2 = (\frac{13}{4})^2 - (\frac{5}{4})^2 = 3^2$.

35. 11·3 cms.

36. Draw AB eastwards of length 10, BC north-east. Let $BC = 4$. Draw CD perp. to AB produced. $AD = 10 + 2\sqrt{2}$, $CD = 2\sqrt{2} \therefore AC^2 = (10 + 2\sqrt{2})^2 + (2\sqrt{2})^2 = 116 + 40\sqrt{2} = 116 + 40 \times 1{\cdot}4142 = 172{\cdot}568 \therefore AC = 13{\cdot}13$.

37. The 4 nearest, the 1 farthest.

38. $\angle AOB = 48\frac{1}{2}°$ approx. Each of the others $= 24\frac{1}{4}°$.

39. PQ becomes the tangent at A.

40. $\angle OAB = 90° - \frac{1}{2}\angle AOB = 75°$, $80°$, $85°$, $87\frac{1}{2}°$, $89\frac{1}{2}°$, $89\frac{3}{4}°$, $89°\ 59\frac{1}{2}'$. When $\angle$AOB becomes zero, the chord becomes a tangent and the $\angle$OAB becomes 90°. Thus the tangent at A is perp. to OA.

41. On AB describe an equilat. △ remote from C, and on AC one remote from B. The intersection of the circumcircles of these △s will give the point required. Prove by III. 13.

42. Let AB be the given str. line. O the centre of the circle. Draw OD perp. to AB. From DB cut off DE equal to ·4 inch. From E draw a perp. to AB, meeting the circle at P. Draw a chord PRQ ∥ to BA, cutting OD at R. PQ is bisected by the perp. OR (III. 3.), and PR = DE (II. 2.) ∴ PQ = ·8 inch.

43. Draw str. lines ∥ to the given str. lines, and at a distance ·7 inch from them. The intersections of these give the reqd. centres. There are 4 positions.

44. In each circle place a chord of length 1 inch. Draw the perps. to these from the respective centres. With these perps. for radii describe two circles. Draw the 4 common tangents to these inner circles (Ex. xxxvi. 1). These are the lines required; for we have drawn them at such a distance from the centres that the intercepted chords = the 1 inch chords previously drawn (III. 10.).

45. Whichever side of the inscribed △ we take, the altitude must be the greatest possible ∴ the vertex must be at the mid. point of the arc; *i.e.* the △ must be isosceles whichever side we take for base ∴ it must be equilateral. The radius = $\frac{2}{3}$ of a median $= \frac{2}{3}$ of a side $\times \frac{\sqrt{3}}{2}$ ∴ a side $= 3\sqrt{3}$ ∴ perimeter $= 9\sqrt{3}$ cms.

EXERCISES XXXIX.

1. Circumference $= \frac{44}{7}$ of radius.

2. Diameter = circumf. $\times \frac{1}{\pi} = 77 \times \frac{7}{22} = \frac{49}{2} = 24\frac{1}{2}$.

3. Distance $= \frac{22}{7} \times 6 \times 6000 = \frac{792000}{7} = 113143$ feet nearly.

4. Number of laps $= \dfrac{1760}{\frac{44}{7} \text{ of } 80} = \dfrac{1760 \times 7}{44 \times 80} = 3\frac{1}{2}$.

5. Distance $= \frac{22}{7} \times 32 \times 2000$ inches $= \frac{22}{7} \times \frac{32 \times 2000}{1760 \times 36}$ miles $= 3\frac{11}{63}$.

6. Distance in 1 minute $= \frac{22}{7} \times 7 \times 240 = 22 \times 240$ feet. Distance in 1 hour $= 22 \times 240 \times 60$ feet = 60 miles.

7. Perimeter of hexagon $= 6r$. Circumf. of circle $= 2\pi r$. Ratios $= \frac{3}{\pi}$.

8. Number of revolutions = distance ÷ circumference of wheel $= 5 \times \frac{22}{7} \times 448 \times 12 \div (\frac{22}{7} \text{ of } 32) = \frac{2240 \times 12}{32} = 840$.

EXERCISES XL.

1. Arc $= \frac{1}{6}$ of circumference $= 4\frac{1}{6}$.

2. Arc $= \frac{54}{360}$ of circumference $= \frac{54}{360} \times \frac{22}{7} \times 14 = 6{\cdot}6$.

3. Arc $= \frac{75}{360} \times \frac{44}{7} \times 21 = 27{\cdot}5$.

4. The $\triangle$ formed by chord and radii is equilateral $\therefore$ arc $= \frac{1}{6}$ circumference $= \frac{1}{6} \times 3{\cdot}1416 \times 170 = 89{\cdot}012 = 89$ to nearest inch.

6. Arc $\frac{22\frac{1}{2}}{360} \times \frac{44}{7} \times 1760 = \frac{1}{16} \times \frac{44}{7} \times 1760 = \frac{44 \times 110}{7} = 691{\cdot}4$ yds.

7. Distance $= \frac{6{\cdot}3}{360} \times 3{\cdot}1416 \times 7925{\cdot}6 = 435{\cdot}7336$.

8. Arc $= 300$, circumference $= 7925{\cdot}6 \times 3{\cdot}14159$. Difference = angle subtended at centre $= \frac{300}{7925{\cdot}6 \times 3{\cdot}14159}$ of $360°$ $= 4{\cdot}34°$.

9. Let ACB be the arc, CDO the bisecting radius, CD the height. $OD = r - 4$, $AO = r$, $AD = 8$ $\therefore$ $r^2 = (r-4)^2 + 8^2$ $\therefore$ $8r = 16 + 64 = 80$ $\therefore$ $r = 10$.

10. Let ACB be the arc, AC half the arc, CD the height, O the centre. $CD = \sqrt{AC^2 - AD^2} = \sqrt{3{\cdot}9^2 - 3{\cdot}6^2} = 1{\cdot}5$ $\therefore$ $DO = r - 1{\cdot}5$ $\therefore$ in $\triangle ADO$ $r^2 = (r - 1{\cdot}5)^2 + 3{\cdot}6^2$ $\therefore$ $3r = 1{\cdot}5^2 + 3{\cdot}6^2 = 15{\cdot}21$ $\therefore$ $r = 5{\cdot}07$.

EXERCISES XLI.

1. Area $= \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} = 38{\cdot}5$ sq. ft.

2. Area $= 3{\cdot}1416 \times \frac{21}{2} \times \frac{21}{2} = 345{\cdot}3614$ sq. ft. $= 345$ sq. ft. 52 sq. in.

3. $\pi r^2 = 3850$ $\therefore$ $r^2 = 3850 \times \frac{7}{22} = 35^2$. Circumference $= 2\pi r$ $= \frac{44}{7} \times 35 = 220$.

4. Area of track $=$ total area $-$ area of grass $= \pi(301^2 - 294^2)$ $= \frac{22}{7} \times (301 - 294)(301 + 294) = \frac{22}{7} \times 7 \times 595 = 13090$.

5. Divide the $\triangle$ into 3 $\triangle$s by joining the incentre to the vertices. The areas of these 3 $\triangle$s are $\frac{1}{2}r \,.\, 5$, $\frac{1}{2}r \,.\, 4$, $\frac{1}{2}r \,.\, 3$. $6r =$ the sum of these $=$ area of whole $\triangle = \frac{1}{2} \times 3 \times 4 = 6 \;\therefore\; r = 1$ $\therefore$ area of circle $= \pi = 3\frac{1}{7}$ sq. ft.

6. $\frac{22}{7}r^2 = 260{\cdot}26 \;\therefore\; r^2 = 260{\cdot}26 \times \frac{7}{22} = 82{\cdot}81 \;\therefore\; r = 9{\cdot}1$.

7. The triangle formed by joining the centres has each side 2 ft. $\therefore$ its area $= \sqrt{3} = 1{\cdot}732$ sq. ft. Each of the 3 sectors is $\frac{1}{6}$ of a circle $\therefore$ together their area $= \frac{1}{2}\pi r^2 = 1{\cdot}5708 \;\therefore$ remaining area $= {\cdot}1612$ sq. ft.

8. $2\pi r = 1 \;\therefore\; r = \dfrac{1}{2\pi} = {\cdot}1592$ ft. Area $= \pi r^2 = 2\pi r \times \frac{1}{2}r = {\cdot}0796$ sq. ft. $= 11{\cdot}46$ sq. in.

9. Area of square $= 200$ (II. 14.); area of circle $= 100\pi = 314{\cdot}16$. Difference $= 114{\cdot}16$.

10. If a $\triangle$ has angles 30°, 60°, 90° it is one-half of an equilateral $\triangle$, and it can be proved by II. 11. that the sides are $2a$, a, $a\sqrt{3}$. The side opposite to the 60° $=$ the side opposite to the 30° $\times \sqrt{3}$. The $\frac{1}{2}$ side of the equilat. $\triangle = \dfrac{\text{perimeter}}{6} = \dfrac{p}{6}$ $\therefore$ the radius of the incircle $= \dfrac{p}{6\sqrt{3}}$. The $\frac{1}{2}$ side of the hexagon $= \dfrac{p}{12} \;\therefore$ the radius $= \dfrac{p\sqrt{3}}{12} \;\therefore$ the radii are as 2 to 3 $\therefore$ the areas are as 4 to 9.

11. Innermost circle $= \dfrac{\pi r^2}{n+1} \;\therefore$ radius $= \dfrac{r}{\sqrt{n+1}}$. Area of 2nd circle $= \dfrac{2\pi r^2}{n+1} \;\therefore$ radius $= \dfrac{r\sqrt{2}}{\sqrt{n+1}}$. Area of 3rd circle $= \dfrac{3\pi r^2}{n+1} \;\therefore$ radius $= \dfrac{r\sqrt{3}}{\sqrt{n+1}}$, and so on.

12. Innermost circle $= \dfrac{\pi r^2}{3} \;\therefore$ radius $= \dfrac{r}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ ft. $= 4\sqrt{3}$ inches $= 6{\cdot}93$. Area of 2nd circle $= \dfrac{2\pi r^2}{3} \;\therefore$ radius $= \dfrac{r\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}$ ft. $= 4\sqrt{6}$ inches $= 9{\cdot}8$.

EXERCISES XLII.

1. $CE^2 = CB^2 - EB^2 = CB^2 - AD^2 = AB^2 + AC^2 - AD^2 = DF^2 + CD^2 = CF^2$.

2. Let OD cut AC at E. $\angle ODA = \angle OAD$ (I. 5.) $= 90° - BAD = 90° - DAE$ $\therefore$ $\angle OEA = 90°$ (I. 22.). $\angle BAD = \angle$ in alt. segment (III. 19.) $= \frac{1}{2}\angle AOD$ (III. 12.) $\therefore$ $\angle AOD = \angle BAC$.

3. Let E be the centre, EF perp. to AB. $FA = \frac{1}{2}AB$ (III. 3.) = a constant. $\angle AEF = \frac{1}{2}\angle AEB = \angle AOB$ (III. 11.) = a constant $\therefore$ AE is of constant length (I. 16.) $\therefore$ EO is also of constant length.

4. $\angle ASR + \angle ARS = 180° - A$ = a constant $\therefore$ arc RPQ + arc PQS = a constant (III. 14.)

5. Let E, F be the centres, EG, FH perps. to AB, AC (BDA being an obtuse $\angle$). $\angle GEA = \frac{1}{2}\angle BEA$ = supplement of $\angle BDA$ (III. 11. and 13.) $= \angle ADC = \frac{1}{2}\angle AFC$ (III. 11.) $= \angle AFH$. $AG = \frac{1}{2}AB = \frac{1}{2}AC = AH$ $\therefore$ $EG = FH$ (I. 16.).

6. $\frac{1}{2}(A - C) = 90° - AEH - (90° - CFG) = \angle CFG - \angle AEH = \angle FEG - \angle EFH$ (III. 18.) $= \angle$ subtending arc FG $- \angle$ subtending arc EH $= \angle$ subtending arc EFG $- \angle$ subtending arc FEH $= \angle EHG - \angle FGH$.

7. Let PAB be the diameter of the larger, PA of the smaller circle. Let the smaller circle roll into a new position in which Q is the point of contact, P′ the new position of P. Join AQ. Let R be the centre in its new position the arc PQ = arc P′Q, for one has rolled on the other. But the radius of the circle is half that of the other $\therefore$ arc P′Q subtends at R twice the angle which the arc PQ subtends at A (p. 198) $\therefore$ $\angle PAQ = \frac{1}{2}\angle P'RQ = \angle P'AQ$ $\therefore$ P′ lies on PAB $\therefore$ as the circle moves the point P traces out the diameter PAB.

8. Let AB be the given str. line. O the centre of the given circle. Make $\angle COD$ equal to twice the given $\angle$, and at C, D draw tangents CE, DE. With centre O and rad. OE, describe a circle cutting AB at F. F is the reqd. pt., for tangents from F will include an $\angle$ equal to $\angle CED$; the chd. of contact will be equal to CD and will $\therefore$ subtend an $\angle$ at the circumference equal to the given $\angle$. The problem is impossible if OE is less than the perp. from O upon the given line.

9. If ABC be the $\triangle$, I the incentre, L, M the excentres opposite to A, B respectively, $\angle IBL = \frac{1}{2}(B + 180° - B) = 90°$. Similarly $\angle ICL = 90°$ $\therefore$ the circle whose diameter is IL passes through B and C. Similarly $\angle LAM = 90°$, and LBM has been proved to be 90° $\therefore$ the circle whose diameter is LM passes through A and B.

10. Let A, B be the pts. Describe a circle on AB as diameter. Cut this at C by a circle with centre B and radius the given length. AC will be the radius of the required circle, BC the tangent.

11. Let L be mid. pt. of AC. Then FL = LC (Ex. xviii. 9.) $\therefore \angle LFC = \angle LCF = 90° - A = \angle FBP$ (Ex. xxxvi. 3.) $\therefore$ FL is a tangent (III. 18.). Similarly for DL. Also BP is a diameter of this circle since D, F are rt. $\angle$s $\therefore$ tangents of B, D are perp. to BD $\therefore$ parallel to AC.

12. Let AC meet the circle at D. In the circle DPBA $\angle DAP = \angle DBP$, and any increase of $\angle DAP$ is accompanied by an equal increase of $\angle DPB$ (III. 12.) $\therefore$ rate of revolution of BP = that of AP. Also by III. 11. rate of revolution of CP = twice that of AP.

13. Let A, B be the centres. In each circle place a chord equal to the given length. Draw AC, BD perp. to these chords. Draw circles with centres A, B and radii AC, BD. Draw a common tangent to these two circles (Ex. xxxvi. 1.) This can be proved to be the required line (III. 10.).

14. Let the incircles of $\triangle$s ABD, ACD touch AD at E, F. $2DE = BD + AD + AB - 2AB = BD + AD - AB$ (Ex. xxxvi. 7.) $2DF = CD + AD - AC$ $\therefore$ $2DE - 2DF = BD - CD - AB + AC = O$ (Ex. xxxvi. 7.) $\therefore$ the two circles touch AD at the same pt. $\therefore$ they touch each other.

15. Draw PO perp. to L and produce it to R so that OR = PO. Draw RQT touching the circle and cutting L at Q. $\angle PQO = \angle RQO$ (I. 4.).

16. $\angle C'OA' = \angle COA$ (I. 3.) $\therefore$ C'A' = CA (III. 14. 15.). Similarly for the other sides $\therefore$ the $\triangle$s are equal in all respects (I. 7.).

17. Let I, I' be the incentre and excentre. These pts. are on the bisector of the $\angle A$. Let IF, I'K be perp. to AB. Draw IL $\parallel$ to AB. $\angle LII' = 45°$ $\therefore$ $\angle LI'I = 45°$ $\therefore$ LI' = LI, *i.e.* $I'K - IF = AK - AF = s - (s - a) = BC$ (Ex. xxxvi. 7.).

18. Let A, B be the centres of smaller and larger circles. Let AB meet the smaller at D, longer circle at E. Let FHG be the common chord, A, E, H, D, B being in one straight line. AH $=\sqrt{13^2-12^2}=5$ $\therefore$ HD $=8$. HB $=\sqrt{15^2-12^2}=9$ $\therefore$ BD $=1$ $\therefore$ DE $=14$.

19. Let A be the centre of one circle, C the point of contact, B the centre of the other circle. Mark two points D on AC at a distance from C equal to the radius of the second circle. Join DB. Make $\angle$DBE equal to $\angle$BDE, the point E being in AC. Let BE meet the second circle in F. EF $=$ EC (I. 6.) $\therefore$ a circle described with centre E and radius EC will be the one required (III. 6.). The two positions of D will give two solutions.

20. Draw LM, LN perp. to AC, AB. Suppose AB greater than AC. Since L is on the bisector of LA, LM $=$ LN. $\triangle$LAC $=$ $\frac{1}{2}$LM . AC $<$ $\frac{1}{2}$LN . AB $\therefore$ $\triangle$LAC $<$ $\triangle$LAB $\therefore$ LC $<$ LB $\therefore$ L lies between D and C. $\angle$C $>$ $\angle$B $\therefore$ $\angle$PAC $<$ PAB $\therefore$ $\angle$PAC $<$ $\frac{1}{2}$A $\therefore$ P lies between L and C $\therefore$ L lies between P and D.

21. See Ex. L. 15.

22. Draw BC, BD perp. to AL, AM. $\angle$BLC $=180^\circ-$ ALB $=$ $\angle$BMD (III. 13.) $\therefore$ LC $=$ MD (I. 16.) $\therefore$ AL $+$ AM $=$ AC $+$ AD $=2$AC $=$ a constant.

23. Take centre C. Let AM meet the circle in R. Draw CTS perp. to BR meeting QP in T. CTS bisects QP and BR (III. 3.). Also TN $=$ SB (II. 2.) $=$ SR $=$ TM $\therefore$ QN $=$ PM.

24. Let O be the centre. In the $\triangle$s POR, QOT, PO $=$ QO, OR $=$ OT $\therefore$ PR $=$ QT and $\angle$OPR $=$ $\angle$OQT (I. 17.) $\therefore$ PR and QT are equal and parallel $\therefore$ PRQT is a parm.

25. Arcs AD, BC together $=$ semicircumference $\therefore$ the $\angle$s subtended by them at the circumference $=$ a rt. $\angle$, *i.e.* $\angle$EBD $+$ $\angle$EDB $=$ a rt. $\angle$ $\therefore$ $\angle$AED is a rt. $\angle$.

26. Distance from vertex to orthocentre $=$ twice distance from circumcentre to base of any $\triangle$ (Ex. xxxvi. 11.) $\therefore$ BR $=$ CQ. Also BR is $\parallel$ to CQ, since both are perp. to AD $\therefore$ RQ is equal and $\parallel$ to BC. Similarly for the other sides $\therefore$ PQRS is equal in all respects to ABCD.

27. Let the bisectors meet in T. $\angle$CPB + $\angle$CQD = $\angle$BCD − $\angle$PBC + $\angle$BCD − $\angle$CDQ = 2 $\angle$BCD − $\angle$ADC − $\angle$ABC = 2 $\angle$BCD − 180° $\therefore$ $\angle$CPT + $\angle$CQT = $\angle$BCD − 90°. But $\angle$CPQ + $\angle$CQP = 180° − $\angle$BCD $\therefore$ by addition $\angle$TPQ + $\angle$TQP = 90° $\therefore$ T = 90°.

28. In $\triangle$ADC, AE is perp. to DC, and CE is perp to AD $\therefore$ DE is perp. to AC (Ex. xxxvi. 3.)

29. Draw diameter AD; produce it to E so that DE = AD. Join EC. BD is the join of mid. pts. of AC, AD $\therefore$ BD is || to CE. But $\angle$ABD is a rt. $\angle$ (III. 17.) $\therefore$ C is a rt. angle $\therefore$ C lies on a circle whose diameter is AE.

30. Produce AP to meet the circumcircle of ABC in Q. $\angle$BCQ = $\angle$BAQ (III. 12.) = 90° − B = $\angle$PCB. Similarly $\angle$CBQ = $\angle$CBP $\therefore$ $\triangle$s PCB, QCB are equal in all respects (I. 16.) $\therefore$ circumcircle of $\triangle$ PCB = circumcircle of $\triangle$ QCB = circle ABC. Similarly for the others

31. Let ABCD be the quadl., AO, BO the bisectors of $\angle$s A, B Draw OE, OF, OH perp. to AB, BC, AD. Then it can be proved by I. 16. that FB = BE, HA = AE, and OE = OF = OH. Draw OG perp. to CD. Suppose OG gr. than OF. Then by II. 11. CG < CF and GD < DH $\therefore$ AB + CD < BC + AD Similarly OG is not less than OF $\therefore$ a circle with centre O passes through E, F, G, H and touches the sides of ABCD.

32. Let OA, OB be fixed radii, P any pt. on the circle, PQ, PR the perps. Let these meet the circle in S, T. R, Q are mid. pts. of PT, PS $\therefore$ RQ = $\frac{1}{2}$TS. But $\angle$P = 180° − $\angle$O = a constant $\therefore$ chord TS is of constant length (III. 14. 15.).

33. Let P be the point, OA, OB the fixed lines. Let the perps PR, PQ be produced to T, S, making RT = PR and QS = PQ. The circle whose centre is O and radius OP passes through T, S. Also TS = 2RQ = a constant, and the $\angle$TPS = supplement of O = a constant $\therefore$ the radius of the circumcircle of TPS is constant, and as it has a fixed centre O, the circle is fixed.

GRAPHS.

EXERCISES XLIII. a.

2. (*a*) mid. pt. (0, 0) the origin; (*b*) mid. pt. (3, 0) a pt. on the axis of x; (*c*) mid. pt. (2, 2); (*d*) mid. pt. $(-4, 4)$.

3. The pts. all lie on a line || to the axis of Y.

4. If A and B are the pts. A lies on OY, B on OX. $\triangle OAB = \frac{1}{2} OA \times OB = 12$ sq. units.

5. The fig. is a rect. whose sides are 6 and 8 units long. Its area $= 6 \times 8 = 48$ sq. units.

6. The ordinates of the first two points are 0 and 12. The abscissae of the other two points are 1·5 and 3·5.

7. Area $= 18$ sq. units.

EXERCISES XLIII. b.

1. (*a*) A str. line || to OY, and at a distance 4 from it on the positive side; (*b*) a str. line || to OX, and at a distance 5 from it on the positive side; (*c*) a str. line || to OY, and at a distance 2 from it on the negative side; (*d*) a str. line || to OX, and at a distance 3 from it on the negative side.

2. (*a*) A str. line thro. the origin, and thro. the point (5, 15); (*b*) a str. line thro. the origin, and thro. the point $(5, -10)$.

3. (*a*) A str. line thro. the origin, and thro. the point (10, 5); (*b*) a str. line thro. the origin, and thro. the point $(10, -5)$.

4. A str. line || to OX, and at a distance 4 from it on the negative side.

5. $y = x + 2$ is a str. line thro. the pts. (0, 2) and (5, 7).

6. $y = x - 2$ is a str. line thro the pts. $(0, -2)$ and (7, 5).

7. A str line thro. the pts. (0, 5) and (5, 10).

8. A str. line thro. the pts. (0, 6) and (5, 11).

9. A str. line thro. the pts. (0, 1) and (5, 11).

10. $y=2x+3$ is a str. line thro. the pts. (0, 3) and (5, 13).

11. $y=4-3x$ is a str. line thro the pts. (0, 4) and (−5, 19).

12. $y=5-6x$ is a str. line thro. the pts. (0, 5) and (1, −1).

13. A str. line thro. the pts. (0, 6) and (1, 8)

14. A str. line thro. the pts (0, 3) and (4, 0).

15. A str. line thro. the pts. (0, −3) and (4, 0).

16. $y=\frac{3x-5}{6}$ is a str. line thro. the pts. $\left(1, -\frac{1}{3}\right)$ and $\left(\frac{5}{3}, 0\right)$.

17. $y=\frac{5-3x}{6}$ is a str. line thro. the pts. $\left(1, \frac{1}{3}\right)$ and $\left(3, -\frac{2}{3}\right)$.

18. A str. line thro. the pts. (5, 3) and (−3, −3).

19. A str. line thro. the pts. (2, 0) and (0, −3).

20. The first line passes thro. the pts. (0, −3), (−5, −13). The second line passes thro. the pts. (0, 7), (14, 0). If these lines are drawn it will be seen that they cut at the pt. (4, 5) $\therefore$ $x=4$, $y=5$ is the reqd. solution.

21. The first line passes thro. the pts. (19, 2), (−2, 8). The second line passes thro. the pts. (7, 1), (17, 7). If these are drawn they will be seen to intersect at the pt. (12, 4).

22. The first line passes thro. the pts. (34, 2), (16, 12). The second line passes thro. the pts. (1, −22), (5, 4). If these are drawn they will be seen to intersect at the pt. (7, 17).

23. The first line passes thro. the pts. (0, 0) and (3, 4). The second line passes thro. the pts (0, 21) and (21, 0). If these are drawn they will be seen to intersect at the pt. (9, 12).

24. The first line passes thro the pts. (4, −5), (6, 9). The second line passes thro. the pts. (−7, 1), (17, 3). They will be seen to intersect at the pt. (5, 2).

25. The first line passes thro. the pts. (15, 0), (0, 15). The second line passes thro. the pts. (5, 0), (0, -5). They will be seen to intersect at the pt. (10, 5).

26. When these pts. are plotted, it will be seen that they all lie on the str. line represented by the equation $y=3x$.

27. *First method.* When these pts. are plotted, it will be seen that they lie in a str. line. If the equation of this line is $ax+by=1$, (0, -5) is on the line $\therefore\ -5b=1\ \therefore\ b=-\frac{1}{5}$, (3, 1) is on the line $\therefore\ 3a+b=1$, $a=\frac{2}{5}$ $\therefore$ the equation reqd. is $y+5=2x$.

Second method. If (x, y) is any pt. on the line, it will be seen from similar $\triangle$s that $\frac{y+5}{x}=2\ \therefore\ y+5=2x$ is the reqd. equation.

28. When the pts. are plotted, it will be seen that they lie in a str. line. If its equation is $ax+by=1$, (0, 4) satisfy the equation $\therefore\ 4b=1$, $b=\frac{1}{4}$, (2, 10) satisfy the equation $\therefore\ 2a+10b=1$, whence $a=-\frac{3}{4}$ $\therefore\ y-4=3x$ is the reqd. equation.

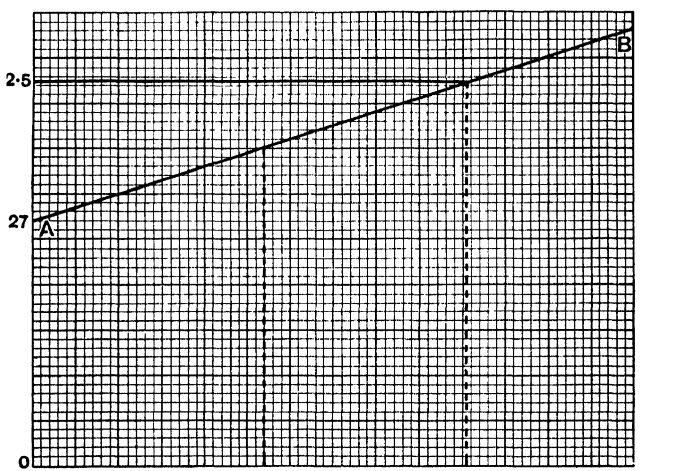
29. If x in. $=y$ cms., $\frac{x}{10}=\frac{y}{25\cdot 4}$. Taking an inch as unit for both x and y values, mark the pt. A whose co-ors. are (10, 25·4). Join OA. OA is the graph of $\frac{x}{10}=\frac{y}{25\cdot 4}$. Take the pt. P on OA whose ordinate is 5·6. Its abscissa will be found to be 2·2 nearly $\therefore$ 5·6 cms. $=$ 2·2 in. nearly. Take the pt. Q on OA whose abscissa is 4·9. Its ordinate will be found to be 11·45 nearly $\therefore$ 4·9 in. $=$ 11·45 cms. nearly.

30. If x cms. $=y$ inches, $\frac{x}{10}=\frac{y}{3\cdot 9}$. Taking an inch as unit for both x and y values, mark the pt. A whose co-ors. are (10, 3·9). Join OA. OA is the graph of $\frac{x}{10}=\frac{y}{3\cdot 9}$. Take the pt. P on OA whose ordinate is 3·6. Estimating the second dec. place, the ordinate will be found to be 9·23 $\therefore$ 3·6 in. $=$ 9·23 cms. Take the pt. Q on OA whose abscissa is 8·6 cms. Estimating the second dec. place, the ordinate will be found to be 3·35 $\therefore$ 8·6 cms. $=$ 3·35 in.

31. Plot the pt. A whose co-ors. are (100, 69). Join OA. OA is the graph whose ordinates correspond to the marks on the paper of max. 69, and whose abscissae correspond to the marks on the paper of max. 100. The abscissae of the pts. whose ordinates are 60, 54, 46, 35, 32, 29, 27, 26, 25, 12 are the marks reqd. These will be found to be (to the nearest integer) 87, 78, 67, 51, 47, 42, 39, 38, 36, 17.

32. 50 articles cost 58 pence. Plot the pt. A whose co-ors. are (58, 50). Join OA. OA is the graph whose abscissae give the price in pence of the number of articles corresponding to its ordinates. The abscissa of the pt. whose ordinate is 23 will be found to be 26·5 $\therefore$ 23 things cost 26·5 pence = 2s. $2\frac{1}{2}$d. The ordinate of the pt. whose abscissa is 36 will be found to be just over 31 $\therefore$ only 31 articles can be obtained for 3s.

33. On paper ruled in inches and tenths of an inch, take OA on a vertical line equal to 2·7 inches, one-tenth of an inch representing one shilling. 800 copies cost $27 + 7 \times 3 = 48$ shillings. Taking an inch horizontally to represent 100 copies,



mark the pt. B whose abscissa is 8 in. (800 copies) and ordinate 4·8 in. (48 shillings). Join AB. The ordinates in the diagram

(which is reduced in printing) give the price in shillings of the number of copies, as shown in the abscissa line. Thus 370 copies cost 35·1s. = 35s. 1d. approx., and for £2. 2s. 6d. we get 615 copies (to the nearest five).

34. Taking one-tenth of an inch horizontally to represent one week, and one-tenth of an inch vertically to represent £1, plot the pt. (52, 120), A. Join OA. The ordinates of pts. on OA give the wages corresponding to the number of weeks represented by the abscissae. The ordinate corresponding to the abscissa 23 will be found to be 53 approx. ∴ the clerk's wages for 23 weeks = £53.

35. Take the ordinate AM of any pt. A on OP, then if PN is the ordinate of P, $\frac{AM}{OM}=\frac{PN}{ON}=\frac{8\cdot66}{10}=\cdot866$ ∴ AM = 0·866 of OM, the dist. of A from OY. The ordinate of the pt. whose abscissa is 3 is 2·60 ∴ 0·866 of 3 = 2·60. The ordinate of the pt. whose abscissa is 6·5 is 5·63 ∴ 0·866 of 6·5 = 5·63. The ordinate of the pt. whose abscissa is 4·8 is 4·16 ∴ 0·866 of 4·8 = 4·16. To find $\frac{1}{0\cdot866}$ of 5, we must read off the abscissa of the pt. whose ordinate is 5. For if Q is that pt. and QK is perp. to OY, $\frac{QK}{OK}=\frac{ON}{PN}$; *i.e.* $\frac{QK}{5}=\frac{10}{8\cdot66}$ ∴ $QK=\frac{1}{0\cdot866}$ of 5. Its value will be found to be 5·77, estimating the second dec. place.

36. If y£ is the cost of x copies, $y=\frac{x}{10}+100$. $\frac{x}{10}+100$ is the reqd. expression. When $x=0$, $y=100$; plot the pt. (0, 1 in.), for 1 in. = £100. When $x=5000$, $y=600$; plot the pt. (5 in., 6 in.), for 5000 copies are represented by 5 inches, and £600 = 6 inches. Join these points by a str. line. The ordinate of any pt. on it gives the price of the no. of copies represented by the corresponding abscissa. 2500 copies = 2·5 inches. The ordinate whose abscissa is 2·5 in. is found to be 3·5 inches = 3·5 × 100 = £350 ∴ 2500 copies cost £350. £525 = 5·25 in. The abscissa of the pt. whose ordinate is 5·25 in. is found to be 4·25 in. = 4·25 × 1000 = 4250 copies ∴ 4250 copies can be obtained for £525.

37. Writing x instead of t, we have to draw the graph of $y=4+3x$. The co-ors. of any pt. on this line give us corre-

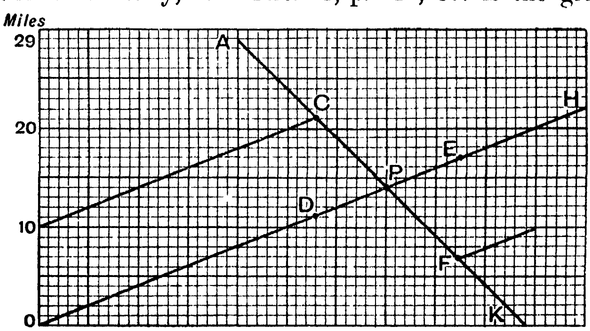
sponding times and velocities. The line passes thro. the pts. (0, 4), (5, 19). Read off the ordinate of the pt. whose abscissa is 3. This is 13. Read off the ordinate of the pt. whose abscissa is 4·5. This is 17·5. Read off the abscissa of the pt. whose ordinate is 11·5. This is 2·5 ∴ 13 and 17·5 ft. per sec. are the velocities reqd and 2·5 secs. the time reqd.

38. If y kilogrammes $= x$ lbs., $\frac{y}{1} = \frac{x}{2 \cdot 2}$ or $\frac{y}{5} = \frac{x}{11}$. Drawing the graph of this equation [a str. line thro. the origin, and thro. the pt. (11, 5)], its ordinates and abscissae give us corresponding numbers of kilogrammes and lbs. From the graph, when $y = 25$, $x = 55$ ∴ 25 kilogrammes = 55 lbs. Similarly, 38 kilogrammes = 84 lbs nearly, 32·5 lbs. = 14·8 kilogrammes, and 38 lbs. = 17·3 kilogrammes.

39. As in the preceding, if x c. ins. $= y$ c. cms. we must draw the graph of $\frac{x}{10} = \frac{y}{164}$ or $\frac{x}{5} = \frac{y}{82}$. The co-ors. of pts. on this line give us corresponding numbers of c. ins. and c. cms. 80 c. cms. = 4 9 c. in. nearly, 40 c. cms. = 2·45 c. in. nearly, 2·5 c. ins. = 41 c. cms.

40. If $x°$ Reaumur $= y°$ Fahr. $\frac{x}{80} = \frac{y - 32}{212 - 32}$, *i.e.* $9x = 4y - 128$. The graph of this equation is the graph reqd. (see Art. 10, p. 212). 60° R. = 167° F., 43° F. = 5° R. nearly.

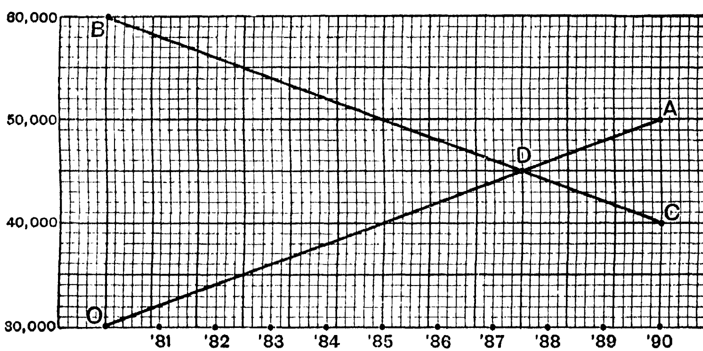
41. Taking 10 units to an hour horizontally, and one unit to a mile vertically, as in Art. 11, p. 214, OH is the graph of



the walker, and AK the graph of the rider. They meet at P, in 3·5 hrs. after noon, viz. at 3.30 p.m. They are 10 m. apart when they are at D and C respectively, *i.e.* in 2·8 hrs. after noon, or at 2.48 p.m. They are also 10 m. apart when they are at E and F respectively, in 4·2 hrs. after noon, *i.e.* at 4.12 p.m.

42. Plot the pt. C whose co-ors. are (15, 100). Join OA. OA is the graph of A's motion, the x values denoting seconds, the y values yards. In OX take OD equal to 3 units. Take also a pt. such that its vertical distance from D is 100, and its horizontal distance from D 12 secs. This will be seen to be the pt. C. Join DC. DC is the graph of B's motion. We thus see that B overtakes A at C, *i.e.* in 15 secs. from A's start and 100 yds. from the starting-point.

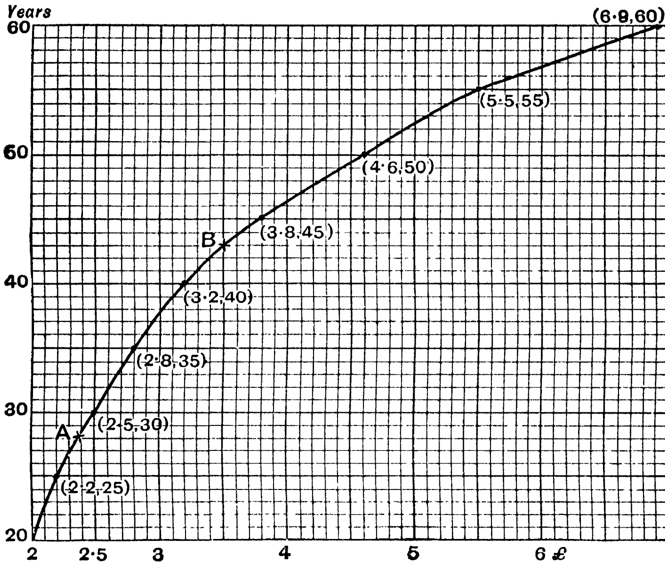
43. Measuring the years along horizontal lines, 6 units to a year, and the populations along vertical lines, 10 units to 10,000, OA in the diagram is the graph showing the growth of



popn. in the first town, and BC the graph showing the popn. in the second town. At D, where OA and BC meet, the popns. are equal, *i.e.* at the end of June, '88.

44. Join the pts. (0, 33), (100, 88). This line is the graph reqd. The abscissae give the scaled marks, the unreduced marks being shown by the ordinates. The scaled marks reqd. are 58, 38, 29.

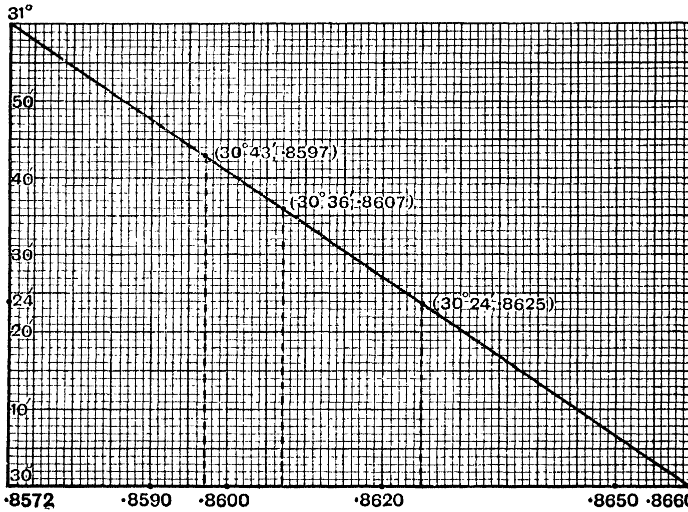
45. The diagram shows the pts. (2, 20), (2·2, 25) (2·5, 30) ... Joining these by an even curve, we have the graph reqd. The abscissae of the pts., whose ordinates are 28 and 43, give the premiums reqd. £24, £35 to the nearest pound.



46. Take an inch (or a centimetre) horizontally to represent ·002, and an inch vertically to represent ·1. Plot the pts. **(17,** 1·2304), (18, 1·2553) and join them. This line is the graph reqd. [*N.B.* The pt. (18, 1·2553) lies 10 inches vertically above (17, 1·2304) and $\frac{24\cdot9}{2}=12\cdot45$ inches horizontally from it.] From the graph we see that the abscissa 1·2395 corresponds to the ordinate 17·36 ∴ log 17·36 = 1·2395. In the same way log **17**·68 = 1·2474. Also the ordinate corresponding to the abscissa 1·2350 is 17·18 ∴ 17·18 is the number whose log is 1·2350. In the same way 17·82 is the number whose log is 1·2508. [The above results are not absolutely true, for the intermediate logs are only approximately proportional to the difference in the numbers.]

47. Measure sines horizontally and degrees vertically. Take one inch horizontally to represent ·001, and one inch vertically to represent 10 minutes. Plot the pts. (50°, ·7660), (51°, ·7771) and join them. [*N.B.* The second pt. lies 6 in. vertically above the first, and 11·1 inches horizontally from it.] The line joining these two pts. is the graph reqd. From it we read off sin 50° 15′ = ·7688, sin 50° 48′ = ·7749, ·7683 = sin 50° 12′, and ·7729 = sin 50° 37′.

48. As the angle increases the cosine diminishes. The diagram now explains itself.



49. Plot the points (22°, ·4040), (23°, ·4245) with the same units as in Example 47, and read off the reqd. values. tan 22° 44′ = ·4190, tan 22° 54′ = ·4224, ·4122 = tan 22° 24′, ·4204 = tan 22° 48′.

50. Use half an inch horizontally to represent one million, and half an inch vertically to represent 10 years. Plot the pts. (8·9, 1801), (10·2, 1811), etc. Join these point to point. It will be seen that the abscissa corresponding to 1837 is 15·1 ∴ 15·1 millions was the popn. in 1837. The year corresponding to the abscissa 24 is 1875.

51. Use paper ruled in cms. and mms., and take one cm. horizontally to denote a year, and one cm. vertically to denote ·1 of a million £. For the first graph plot the pts. (1884, 2·51), (1885, 2·47), etc. For the second graph plot the pts. (1884, 1·29), (1885, 1·35), etc. It will be seen from the two papers that, usually, as the total expenditure diminishes the salaries of officials increase.

52. Use paper ruled in inches and tenths of inches; take half an inch horizontally to denote a month, and one inch vertically to denote an inch of rainfall. Plot the pts. estimating the second dec. place.

53. Using paper ruled in inches and tenths of an inch (or in cms. and mms.), take one-tenth horizontally to denote a month, and one-tenth vertically to denote a penny. Plot the pts. (1891, 45), (1892, 40), etc., and join them by an even curve. This is the reqd. graph. Price of silver on May 1st, 1895, 30·4 pence.

54. Take one-tenth of an inch horizontally to represent one foot, and one inch vertically to represent one second. Plot the pts. (2, 1), (6, 2), (12, 3), (20, 4), and so on. [*N.B.* The total space in 4 secs. $= 2 + 4 + 6 + 8 = 20$.] Join the pts. by an even curve and we have the graph. The abscissa corresponding to the 4·2 ordinate is 22 nearly $\therefore$ the body describes 22 ft. in 4·2 secs. In the same way we see that the body describes 62 ft. (nearly) in 7·4 secs. The ordinate corresponding to the abscissa 15 is 3·4 $\therefore$ the body takes 3·4 secs. to describe 15 feet.

55. Take one inch horizontally to represent one inch of the string, and one inch vertically to represent one pound. Plot the pts. (7·7, ·6), (8·0, 1·2), (8·4, 2·0), (8·8, 2·8), (9·0, 3·2). When we examine these we see they lie in a straight line. This line is the graph reqd. The ordinate corresponding to the abscissa 10 is 5·2 $\therefore$ 5·2 lbs. will stretch the string to 10 inches. The abscissa corresponding to the ordinate 2·25 is 8·5 $\therefore$ when the wt. is 2·25 lbs. the stretched length is 8·5 inches. The unstretched length is 7·4 in.

56. Use paper ruled in cms. and mms.; take 1 mm. horizontally to represent one degree, and 1 mm. vertically to represent ·01 of a radian. Plot the pts. (0, 0), (15, ·26), etc. Join them.

The graph will be a str. line. From it we see that 40 degrees $= \cdot 7$ radians, 70 degrees $= 1 \cdot 22$ radians, $\cdot 64$ radians $= 37°$, and $\cdot 86$ radians $= 49°$.

57. Take one-tenth of an inch horizontally to denote 10°, and one-tenth of an inch vertically to represent $\cdot 1$. Estimating carefully the second dec. place, plot the pts. (0, 0), (15, $\cdot 26$), (30, $\cdot 5$), etc. To obtain the graph from 90° to 180°, use the facts that $\sin 120° = \sin 60°$, $\sin 135° = \sin 45°$, $\sin 150° = \sin 30°$, $\sin 180° = 0$. The graph from 0° to $-180°$ may be similarly obtained, remembering that $\sin(-30°) = -\sin 30°$, and so on.

58. Use the same units as in Example 57, and we obtain the graph in a similar manner.

59. With the same units as in the preceding two examples plot the points. We shall see that the vertical line through the 90° pt. is an asymptote to the graph (see Art. 22.).

EXERCISES XLIV. a.

1. A circle, centre at the origin, rad. 6 units (Art. 12).

2. A point. The origin.

3. A circle, centre at the origin, rad. 7 units (Art. 12).

4. A circle, centre at the origin, rad. 9 units (Art. 12).

5. The equation may be written $(x+4)^2 + (y-4)^2 = 32$ $\therefore$ the graph is a circle, centre at the pt. $(-4, 4)$, rad. $4\sqrt{2}$ (Art. 14).

6. The equation may be written $(x-4)^2 + (y-3)^2 = 25$ $\therefore$ the graph is a circle, centre at the pt. (4, 3), rad. 5 (Art. 14).

7. A circle, centre at (3, 4), rad. 6 (Art. 14).

8. A circle, centre at (1, 2), rad. 6 (Art. 14).

9. A circle, centre at $(-2, 3)$, rad. 5 (Art. 14).

10. A circle, centre $(3, -3)$, rad. 4 (Art. 14).

11. $y = \sqrt{15 - 2x - x^2}$ $\therefore$ $y^2 = 15 - 2x - x^2$; $x^2 + 2x + y^2 = 15$; $(x+1)^2 + y^2 = 16$ $\therefore$ the graph is a circle, centre at $(-1, 0)$, rad. 4 (Art. 14).

12. $y=\sqrt{21+4x-x^2}$; $y^2=21+4x-x^2$; $x^2-4x+y^2=21$; $(x-2)^2+y^2=25$ $\therefore$ the graph is a circle, centre (2, 0), rad. 5 (Art. 14).

13. $y=\sqrt{15+2x-x^2}$; $y^2=15+2x-x^2$; $x^2-2x+y^2=15$; $(x-1)^2+y^2=16$ $\therefore$ the graph is a circle, centre at (1, 0), rad. 4 (Art. 14).

14. $y=\sqrt{14x-x^2-13}$; $y^2=14x-x^2-13$; $x^2-14x+y^2=-13$; $(x-7)^2+y^2=36$ $\therefore$ the graph is a circle, centre at (7, 0), rad. 6 (Art. 14).

15. The graph of $x^2+y^2=36$ is a circle, centre at the origin, rad. 6 (Art. 12). $x^2+y^2-8x-20=0$ may be written $(x-4)^2+y^2=36$ $\therefore$ its graph is a circle, centre at (4, 0), rad. 6 (Art. 14). Drawing these circles, they will be seen to meet at the pts (2, 5·66), (2, −5·66) $\therefore$ $x=2$, $y=5·66$, and $x=2$, $y=-5·66$ are the reqd. solutions. [Half an inch or one inch should be taken as the unit.]

16. When

$y=0$	± 2	± 3	± 4	± 5	...
$x=0$	2	4·5	8	12·5	...

Points on the graph are given by the above table. Joining them, we have a parabola (see Art. 16).

17. $4x=y^2+8$. When

$y=0$	± 1	± 2	± 4	± 6	...
$x=2$	2·25	3	6	11	...

Points on the graph are given by the above table. Joining them, we have a parabola.

18. $y=2\sqrt{x-4}$, $y^2=4(x-4)$, $4x=y^2+16$. When

$y=0$	± 1	± 2	± 4	± 8	...
$x=4$	4·25	5	8	20	...

Points on the graph are given by the above table. Joining them we have a parabola.

19. $y=4\sqrt{x+4}$, $y^2=16(x+4)$, $16x=y^2-64$, $x=\frac{y^2-64}{16}$.

When

$y=0$	± 2	± 4	± 6	± 8	± 10	...
$x=-4$	$-3{\cdot}75$	-3	$-1{\cdot}75$	0	$2{\cdot}25$	...

Points on the graph are given by the above table. Joining them, we have a parabola. [Four-tenths of an inch will be a suitable unit for the x values.]

20. When

$y=0$	1	2	3	4	5	..
$x={\cdot}25$	0	${\cdot}25$	1	$2{\cdot}25$	4	...

When

$y=-1$	-2	-3	-4	-5	...
$x=1$	$2{\cdot}25$	4	$6{\cdot}25$	9	...

Points on the graph are given by the above tables. Joining them, we have a parabola. [Four-tenths of an inch will be a suitable unit for the x values.]

21. When

$x=0$	± 1	± 2	± 3	± 4	± 5	...
$y=0$	${\cdot}25$	1	$2{\cdot}25$	4	$6{\cdot}25$	...

Points on the graph are given by the above table. Joining them, we have a parabola. [Four-tenths of an inch will be a suitable unit.]

22. When

$x=0$	± 1	± 2	± 3	± 4	± 5	...
$y=2$	$2{\cdot}25$	3	$4{\cdot}25$	6	$8{\cdot}25$	...

Points on the graph are given by the above table. Joining them, we have a parabola. [Four-tenths of an inch will be a suitable unit.]

23 When

$y=0$	± 1	± 2	± 3	± 4	± 5	...
$x=0$	$-\cdot 25$	-1	$-2\cdot 25$	-4	$-6\cdot 25$	...

Points on the graph are given by the above table. Joining them, we have a parabola. [Four-tenths of an inch will be a suitable unit.]

24. When

$y=0$	± 1	± 2	± 3	± 4	± 5	...
$x=2$	$1\cdot 75$	1	$-\cdot 25$	-2	$-4\cdot 25$	...

Points on the graph are given by the above table. Joining them, we have a parabola. [Four tenths of an inch will be a suitable unit.]

25. When

$x=$	0	± 1	± 2	± 3	± 4	± 5	± 6	...
$y=$	2	$1\cdot 75$	1	$-\cdot 25$	-2	$-4\cdot 25$	-7	..

Points on the graph are given by the above table. Joining them, we have a parabola. [Four-tenths of an inch is a suitable unit.]

26. $4y=(x-1)^2$. When

$x=0$	1	2	3	4	5	6	...
$y=\cdot 25$	0	$\cdot 25$	1	$2\cdot 25$	4	$6\cdot 25$	...

When

$x=-1$	-2	-3	-4	-5	...
$y=1$	$2\cdot 25$	4	$6\cdot 25$	9	...

Points on the graphs are given by the above tables. Joining them, we have a parabola.

27. When

$x=$	0	1	2	3	4	5	6
$x^2=$	0	1	4	9	16	25	36
$-6x+1=$	1	-5	-11	-17	-23	-29	-35
$y=$	1	-4	-7	-8	-7	-4	1

Plot the pts. $(0, 1)$, $(1, -4)$, $(2, -7)$, $(3, -8)$, $(4, -7)$, $(5, -4)$, $(6, 1)$, and join them by an even curve. $x^2-6x+1=0$ when $y=0$, *i.e.* when the graph cuts the axis of x. From the graph we see that at these pts. $x=5{\cdot}8$, or $\cdot 2$ $\therefore$ $5{\cdot}8$ and $\cdot 2$ are approximate roots of the equation. Also we see that -8 is the least value of y $\therefore$ the minimum value of x^2-6x^2+1 is -8. [Use an inch for the x unit, and half an inch for the y unit.]

28. When

$x=$	-2	-1	0	1	2	3
$4x^2=$	16	4	0	4	16	36
$4x+15=$	7	11	15	19	23	27
$y=4x^2-4x-15=$	9	-7	-15	-15	-7	9

Plot the pts. $(-2, 9)$, $(-1, -7)$, $(0, -15)$, $(1, -15)$, $(2, -7)$, $(3, 9)$. Join them by an even curve. This gives the reqd. portion of the graph. The roots of $4x^2-4x-15=0$ are given by the x values when the graph cuts the axis of x. From the graph we see that these are $-1{\cdot}5$, $2{\cdot}5$. [Use an inch as the x unit, and one-tenth of an inch as the y unit.] From the symmetry of the graph we see that y is a minimum when $x=\cdot 5$, *i.e.* when $y=4(\cdot 5)^2-4\times\cdot 5-15=-16$.

29. The graph of $x^2+y^2=25$ is a circle, rad. 5. Describe it, using half-inch units. $(0, -5)$, $(4, 7)$ are points on the graph

of $y=3x-5$. Join them by a str. line. This is the graph of $y=3x-5$. We see that the str. line and circle cut at the pts. $(0, -5)$, $(3, 4)$ $\therefore x=0$ or 3, $y=-5$ or 4 are the roots reqd.

30. Use a centimetre as the x unit, and half a centimetre as the y unit. Trace the graph of $y=x^2$ (Art. 16), and the graph of $8y-10x-75=0$. [$(-7{\cdot}5, 0)(-3{\cdot}5, 5)$ are pts. on this line.] The abscissae of the pts. where the graphs meet give the roots of the equation (Art. 18). These will be found to be $-2{\cdot}5$ and $3{\cdot}75$.

31. $x^2-6x+5=0$, *i.e.* $(x-1)(x-5)=0$; $x-1=0$, or $x-5=0$ $\therefore$ the graph is two str. lines $\parallel$ to OY and distant 1 and 5 units from it on the positive side.

32. $y^2+5y+6=0$; $(y+2)(y+3)=0$; $y+2=0$, or $y+3=0$ $\therefore$ the graph is two str. lines $\parallel$ to OX and distant 2 and 3 units from it on the negative side.

33. $x^2+x-6=0$; $(x+3)(x-2)=0$; $x+3=0$, or $x-2=0$ $\therefore$ the graph is two str. lines $\parallel$ to OY; the first at a distance 3 units from it on the negative side, the other at a distance 2 units from it on the positive side.

34. $y^2-3y-28=0$; $(y-7)(y+4)=0$; $y-7=0$, or $y+4=0$ $\therefore$ the graph is two str. lines $\parallel$ to OX; the first at a distance 7 units from it on the positive side, the other at a distance 4 units from it on the negative side.

35. The given equation may be written $(x+2y)(x+3y)=0$ $\therefore$ $x+2y=0$, or $x+3y=0$; *i.e.* the graph is two str. lines through the origin (Art. 15). The first passes thro. the pt. $(6, -3)$. The second passes thro. the pt. $(6, -2)$.

36. The given equation may be written $(2x+y)^2=0$ $\therefore$ the graph is two *coincident* str. lines, each represented by the equation $2x+y=0$. This line is thro. the origin and thro. the pt. $(4, -8)$.

37. Draw the graph of $y=x^2$, using an inch as the x unit, one-tenth of an inch as the y unit (Art. 16). With the same units draw the graph of $y=3x+6$. [It passes thro. $(0, 6)$ and $(-2, 0)$]. The abscissae of the two pts. where these graphs meet give us the roots reqd. They are seen to be $4{\cdot}4$ and $-1{\cdot}4$

38. Draw the graphs of $y = x^2$ and $y + 4x = 8$ with the same units as in the preceding example. The roots, given by the abscissae of the pts. where these meet, will be seen to be $-1{\cdot}46$ and $-5{\cdot}46$.

39. With the same units, draw the graph of $y = x^2$, and the graph of $y - x - 20 = 0$. [$(-6, 14)$, $(0, 20)$ are pts. on the str. line.] The abscissae of the pts. where these graphs meet give the reqd. roots. They will be found to be -4 and 5. From the graph we see that $x^2 - x - 20$ is negative as long as the pt. whose abscissa in x lies between the pts. where the graphs meet, *i.e.* as long as x is between -4 and 5 (Art. 18).

40. Use one centimetre for x and y units, and draw the graphs of $x^2 + y^2 = 25$ and $x - 2y + 2 = 0$. The pts. where these graphs meet will be found to be $(4, 3)$ and $(-4{\cdot}8, -1{\cdot}4)$ $\therefore$ $x = 4$ or $-4{\cdot}8$, $y = 3$ or $-1{\cdot}4$ are the reqd. roots.

41. Use one inch for the x unit, and one-tenth of an inch for the y unit. Trace the graphs of $y = x^2$ and $x + 6 - 2y = 0$. The abscissae of the pts. where these graphs meet give the reqd. roots. They will be seen to be 2 and $-1{\cdot}5$. As in Art. 16, we see that the expression $x + 6 - 2x^2$ is positive as long as x lies between 2 and $-1{\cdot}5$.

42. Trace the graph of $y = 2x^2 - x - 6$. When

$x = 0$	1	2	3	4
$2x^2 = 0$	2	8	18	32
$x + 6 = 6$	7	8	9	10
$y = 2x^2 - x - 6 = -6$	-5	0	9	22

When

$x =$	-1	-2	-3
$2x^2 =$	2	8	18
$x + 6 =$	5	4	3
$y = 2x^2 - x - 6 =$	-3	4	15

Plotting the points (0, -6), (1, -5), (2, 0), (3, 9), (4, 32), (-1, -3), (-2, 4), (-3, 15), and joining them, we have the graph. From the figure we see that the least value of y, *i.e.* of $2x^2-x-6$, is $-6\cdot1$ approx.

43. Use the formula $s=ut+\frac{1}{2}ft^2$. In 1 second the first particle rises $128-16=112$ ft. In 2 seconds the first particle rises $256-64=192$ ft. In 3 seconds the first particle rises $384-144=240$ ft. In 4 seconds the first particle rises $512-256=256$ ft. In 1 second the second particle falls 16 ft. In 2 seconds the second particle falls $16\times4=64$ ft. In 3 seconds the second particle falls $16\times9=144$ ft. In 4 seconds the second particle falls $16\times16=256$ ft. Take A as the starting pt. of the first, and B, 256 units vertically above it, as the starting pt. of the second particle. Measuring the seconds of time horizontally from A and B, and using a fairly large time unit, say 2 inches, plot the pts. (1, 112), (2, 192), (3, 240), (4, 256) for the first particle, and join them by an even curve. The ordinates must be measured upwards from A to B. Plot the pts. (1, 16), (2, 64), (3, 144), (4, 256) for the second particle, measuring the times horizontally, and the distances *vertically downwards* from B. The time of the pt. where these graphs meet gives us the time of the meeting of the particles. It will be found to be 2 secs. To find when they are 160 ft. apart, mark off a length of 160 units on a straight edge of paper, and move it || to AB until the vertical dist. between pts. on the graphs is equal to this marked distance. Read off the time from the graph. It will be found to be $\frac{3}{4}$ sec. or $3\frac{1}{4}$ sec.

44. Use the formula $v^2=2gs$. When

$v=8$	16	24	32	40	48	56	64	72	80	88	96
$s=1$	4	9	16	25	36	49	64	81	100	121	144

Use one mm for both v and s units. Measure s vertically downwards and v horizontally. Plot the pts. (8, 1), (16, 4), (24, 9), etc., and join them by an even curve. This is the graph reqd. When $s=124$, we see that $v=89$ approx. $\therefore$ the velocity of the body when it has fallen 124 ft. is 89 ft. per sec. approx.

45. Use the formula $s = \frac{1}{2}ft^2$ for the motion of the second particle. When

$t=$	1	2	3	4	5	6	...	secs.
$s=$	4	16	36	64	100	144	...	feet.

Plot the pts. (1, 4), (2, 16), etc., measuring times horizontally with an inch unit, and spaces vertically upwards with one-tenth of an inch as unit. Join them, and we have the graph of the second particle. Take a point 48 units vertically above the starting pt. of the second particle, and measuring lines horizontally as before, and spaces vertically *downwards*, draw the graph of the first particle. This is a str. line, thro. the pts (1, 4), (2, 8), (3, 12), etc. The time given by the point where the graphs meet gives us the time of meeting. It will be seen to be 3 secs. To find when they are 33 ft. apart, thro. a pt. 33 units vertically below the starting pt. of the first particle, draw a str. line || to its graph. The line of the pt. where this meets the graph of the second particle gives us the time reqd. It will be found to be 1·5 secs.

EXERCISES XLIV. b.

1. Beginning at the bottom of the sheet from the left, mark the inches horizontally 5, 5·2, 5·4, etc., and vertically 25, 27, 29, etc. Mark the points (5, 25), (6, 36). Mark also $(5{\cdot}2, 5{\cdot}2^2)$, $(5{\cdot}4, 5{\cdot}4^2)$, $(5{\cdot}6, 5{\cdot}6^2)$, *i.e.* (5·2, 27·04), (5·4, 29·16), (5·6, 31·36). Connect these points by a smooth curve, and read off the abscissa corresponding to the ordinate whose square root is wanted. Thus $\sqrt{31} = 5{\cdot}57$, $\sqrt{28} = 5{\cdot}29$, $\sqrt{29{\cdot}6} = 5{\cdot}44$, $\sqrt{31{\cdot}3} = 5{\cdot}6$.

2. Mark the inches horizontally 5, 5·2, 5·4, etc., and vertically 125, 135, 145, etc. Mark the points (5, 125), (6, 216). Mark also $(5{\cdot}4, 5{\cdot}4^3)$, $(5{\cdot}6, 5{\cdot}6^3)$, *i.e.* (5·4, 157·46), (5·6, 175·6). Connect these points by a smooth curve Read off the abscissae corresponding to ordinates 144, 198. Thus $\sqrt[3]{144} = 5{\cdot}24$, $\sqrt[3]{198} = 5{\cdot}83$.

3. For the graph of $y=x^3$ we have

$x=0$	·5	1	1·5
$y=0$	·125	1	3·375

Take 2 inches for the unit, and, by means of the values given above, draw the graph from (0, 0) to (1·5, 3·375). The graph of $y=2x-1$ is a str. line passing through (2, 3) and (·5, 0). The values of x at the intersections are ·62 and 1. These are the positive values of x which make x^3 equal to $2x-1$, *i.e.* they are positive solutions of $x^3-2x+1=0$.

4. Mark at the inches horizontally 6, 6·2, 6·4, etc., and vertically 38, 40, 42, etc., beginning from an intersection near the bottom left-hand corner. Mark the points (6, 36), (7, 49). Mark also (6·3, $6{\cdot}3^2$), (6·6, $6{\cdot}6^2$), *i.e.* (6·3, 39·69), (6·6, 43·56). Connect these points by a smooth curve and read off the abscissa corresponding to any ordinate whose sq. rt. is required. Thus $\sqrt{39{\cdot}4}=6{\cdot}28$, and $\sqrt{46{\cdot}7}=6{\cdot}83$.

5. It passes through (1, 1), (2, $\frac{1}{2}$), (3, $\frac{1}{3}$), etc., ($\frac{1}{2}$, 2), ($\frac{1}{3}$, 3), etc., (-1, -1), (-2, $-\frac{1}{2}$), (-3, $-\frac{1}{3}$), etc., ($-\frac{1}{2}$, -2), ($-\frac{1}{3}$, -3), etc. A rectangular hyperbola with the axes of x and y for asymptotes. (See § 22.)

6. A rectangular hyperbola passing through the points (2, 2), (4, 1), (5, ·8), etc., (1, 4), (·8, 5), etc., (-2, -2), (-1, -4), ($-$·8, -5), etc., (-4, -1), (-5, $-$·8), etc. (See § 22.)

7. and **8.** Determine points as in Example 6. A rectangular hyperbola in each case.

9. When $y=0$, ±1, ±2, ±4, ±5, ±6, ±7, ±8; $x=\pm1$, $\pm$·99, $\pm$·97, $\pm$·87, $\pm$·78, $\pm$·66, $\pm$·48, 0. Ellipse, centre (0, 0), semi-axes along the axes of x, y, and equal to 1 and 8.

10. Ellipse, centre (0, 0), semi-axes along the axes of x, y, and equal to 5 and 1.

11. When $y=0$, ±2, ±4, ±6, ±8, etc., $x=\pm8$, $\pm8{\cdot}2$, $\pm8{\cdot}9$, ±10, $\pm11{\cdot}3$, etc. A rectangular hyperbola, centre (0, 0).

12. As Question 11.

13. When $x = \pm 1$, $\pm 1{\cdot}5$, ± 2, etc., $y = 0$, $\pm 8{\cdot}9$, $\pm 13{\cdot}9$, etc. Hyperbola, centre (0, 0).

14. $\frac{x^2}{9} + \frac{y^2}{16} = 1$. Ellipse, centre (0, 0), semi-axes 3 along the axis of x, 4 along the axis of y.

$$\left.\begin{array}{l}\textbf{15.}\ \text{Hyperbola}\\ \textbf{16.}\ \text{Hyperbola}\end{array}\right\}\text{as in 13.}$$

17. $6x^2 - 5xy + y^2 = 0$, *i.e.* $(3x - y)(2x - y) = 0$. The equation is satisfied by every point on the str. line $3x - y = 0$, and every point on the str. line $2x - y = 0$ $\therefore$ the graph is 2 str. lines passing through (0, 0), one of them going through (1, 3), the other through (1, 2).

18. The two str lines $2y + x = 0$, $2y - x = 0$.

19. The equation is satisfied by every point along the str. line $x = 0$, and every pt. along the str. line $y = 0$: *i.e.* the graph is the two axes.

20. $x^2 + y^2 = 0$. This is not satisfied by any point except (0, 0) $\therefore$ it represents the origin. In fact it is a circle whose centre is (0, 0) and radius zero.

21. Two str. lines, one || to the axis of y at a distance 3 from it, the other || to the axis of x at a distance 4 from it.

22. This is satisfied only by $x - 3 = 0$, and $y - 4 = 0$ simultaneously. It represents the point (3, 4).

23. A rectangular hyperbola.

When

$x = 3$	2	1	0	-1
$y = {\cdot}5$	1	∞	-1	$-{\cdot}5$

etc.

24. When

$x = 5$	4	3	2	1	0	-1
$y = 12$	6	2	0	0	2	6

a parabola with vertex at $(1{\cdot}5, -{\cdot}25)$.

25. When

$x=$	-1	0	1	2	3	4	5	6
$y=$	-75	-32	-9	0	1	0	3	16

The curve cuts the axis of y where $y=-32$, rises steeply through $(1, -9)$ to $(3, 1)$, cutting the axis of x at $(2, 0)$, bends down towards the axis of x, which it touches at $(4, 0)$ and rises again steeply to an infinite distance. It also goes to infinity in a negative direction. By the form $y=(x-2)(x-4)(x-4)$, it is clear that the axis of x is cut at $(2, 0)$, and cut in 2 coincident pts. (*i.e.* touched) at $(4, 0)$.

26. When $x=0$, $\cdot 5$, 1, 1 5, 2, 3, 4, $y=0$, $\cdot 25$, 2, $3\cdot 38$, 16, 54, 128. These points enable us to draw the graph in the 1st quadrant, and the rest is in the 3rd quadrant symmetrically situated to this part, as may be seen by putting $-x$, $-y$, for x, y.

27. In the figure of § 17, if we were to change the sign of every abscissa without altering the ordinate we should get the graph of $y=-x^3$. In fact the graph of $y=-x^3$ may be seen by holding up to the light the graph of $y=x^3$, looking at it through the paper from the back.

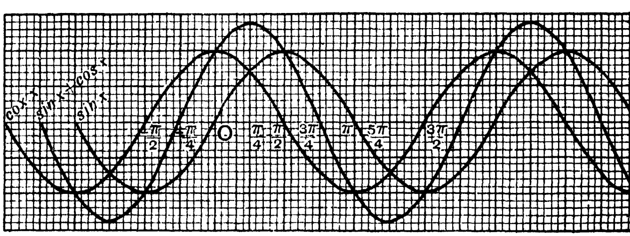
28. Any value of y gives two equal and opposite values of x $\therefore$ the curve is symmetrical with regard to the axis of y, but not with regard to the axis of x. No negative values of y are possible. Thus the curve lies on the upper side of the axis of x, touching it at the origin. Points $(0, 0)$, $(1, 1)$, $(\frac{1}{2}, \frac{1}{16})$, $(\frac{3}{2}, \frac{81}{16})$, $(2, 16)$ enable us to draw it.

29. $y=(x-1)(x-2)^2$. When $x=-2$, -1, 0, 1, $\frac{4}{3}$, $\frac{3}{2}$, 2, 3, 4, 5, $y=-48$, -18, -2, 0, $\frac{4}{27}$, $\frac{1}{8}$, 0, 2, 12, 36, the graph ascends steeply from $(-2, -48)$ to $(1, 0)$, where it cuts the axis of x, then by values of x near to $\frac{4}{3}$ it will be found to turn downwards at the point $(\frac{4}{3}, \frac{4}{27})$. It touches the axis of x at $(2, 0)$ and rises to an infinite distance as indicated by the points given.

30. $y=x^3-4x+1$. When $x=-3$, -2, -1, 0, $\cdot 5$, 1, $1\cdot 5$, 2, 3, $y=-14$, 1, 4, 1, $-\cdot 88$, -2, $\cdot 375$, 1, 16. The diagram drawn by means of these pts. shows that the function vanishes for

three values of x lying respectively between -3 and -2, between 0 and 1, and between 1 and 2.

31. The unit for x may be taken the same as that for y; but for this question it is more convenient to mark the inches along the axis of x as $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, etc.; those on the axis of y as 1, 2, etc. The pts. $(0, 0)$, $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$, $\left(\frac{\pi}{2}, 1\right)$, etc., lie on the graph of $\sin x$; and by continuing in this way we get the graph, which extends similarly on both sides of the origin. By moving the curve back through one inch $\left(\frac{\pi}{2}\right)$ along the axis of x we get the graph of $\cos x$. By taking for each value of x an ordinate equal to the algebraic sum of the corresponding ordinates of $\sin x$ and $\cos x$ we get the graph of $\sin x + \cos x$. This curve cuts the axis of x at $-\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, etc. Thus the general solution of the equation $\sin x + \cos x = 0$ is $x = n\pi - \frac{\pi}{4}$.



32. When

$x=0°$	1°	2°	3°	4°	5°	5·8°	6°	7°	8°	9°
$\tan 10x - 2\tan 9x + 1 = 1$	·86	·72	·56	·39	·19	·02	$-$·02	$-$·18	·52	∞

From these the graph may be drawn. It crosses the axis of x where $x = 5\ 9$ approximately $\therefore$ $\tan 10x - 2\tan 9x + 1$ vanishes when $x = 5·9°$. The graph crosses also at 7·6 $\therefore$ the expression vanishes when $x = 7·6°$.

33. Beginning at the bottom left-hand point, mark the vertical inches as 39°, 41°, 43°, etc., and the horizontal inches 0, 4 p.m., 8 a.m., etc Join the points indicated, and read off the temperature for 3 p.m. Result 52·1°.

34. Take the tenths of an inch along the axis of x to represent minutes of time, those along the axis of y to represent minute divisions on the clock face The long hand travels 60 divisions in 60 minutes ∴ the line joining (0, 0) to (60, 60) will represent its motion. The short hand starts 5 divisions ahead (at 1 o'clock), but goes only 5 divisions in 60 minutes ∴ the line joining (0, 5) to (60, 10) represents its motion. The intersection of these shows (by the abscissa) how many minutes after 1 they are together. A line drawn parallel to the 2nd graph and at a vertical distance 30 from it [*i.e.* a line joining (0, 35) to (60, 40)] will, by its intersection with the 1st graph, show the time at which the hands are opposite. A line drawn parallel to the graph of the short hand at a vertical distance 10 from it will, by intersecting the graph of the long hand, show at how many minutes past 1 o'c. the hands are 10 divisions apart. Similarly for 25 divisions and for 15 divisions apart (*i e.* for hands at right angles). Results to the nearest minute : (*a*), (1) 5 minutes past 1, (2) 38, (3) 16, (4) 22, (5) 33. The same method will do for the times between 4 and 5 o'clock, but the 2nd graph in this case is the str. line from (0, 20) to (60, 25) since the hour hand has a start of 20 divisions. (*b*) Results, (1) 22 minutes past 4, (2) 55, (3) 11 and 33, (4) 5 and 38, (5) 49. (*c*) Results, (1) 27 minutes past 5, (2) at 6 o'clock only, (3) 16 and 38, (4) 11 and 44, (5) 55. (*d*) Results, (1) 44, (2) 11, (3) 33 and 55, (4) 27, (5) 16.

35. Take the directrix for axis of y, O the origin ; and along the axis of x mark off OS 1 inch. On the axis of x take any point N. With centre S and radius ON describe a circle cutting at P (above and below the axis) the ordinate through N. P is a point on the parabola. Similarly any number of points may be found.

36. Take the directrix for axis of y, O the origin ; and along the axis of x mark off OS 2 inches. Join the points (0, 0) and (10, 7), and let this line cut the ordinate through

any pt. N (on the axis of x) at Q. With centre S and radius NQ describe a circle cutting the ordinate through N at P. SP = NQ = ON × ·7 = e. ON, where e = eccentricity ∴ P is a point on the conic. Similarly other points may be found.

37. Take X for origin, the directrix for axis of y. Let XS be 1 inch. Join (0, 0) to (1, 1·5) and produce this line to meet at Q the ordinate through any point N on the axis of x. With centre S and radius NQ describe a circle cutting at P the ordinate through N. SP = NQ = ON × 1·5 = e. ON, where e = eccentricity ∴ P is a point on the required hyperbola Similarly other points may be found.

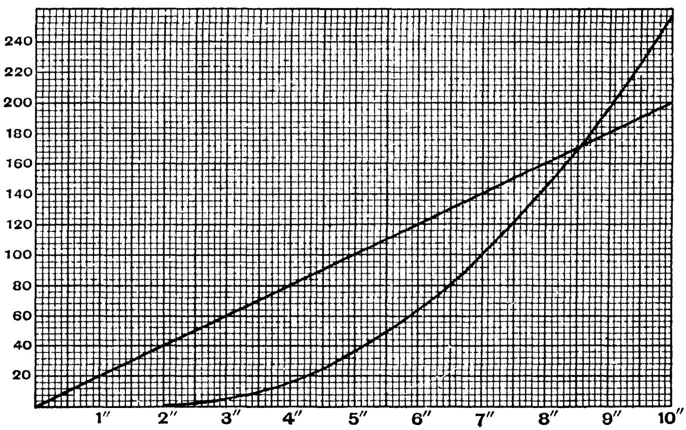
38. Take the axis of y for directrix, S the focus on the axis of x, X the origin. Join (0, 0) to (1, 1) and let this line cut at Q the ordinate through any pt N on the axis of x. With centre S and radius XQ describe a circle cutting the ordinate NQ at P. SP = XN . $\sqrt{2}$ = e . XN [since the eccentricity of a rectangular hyperbola is $\sqrt{2}$] ∴ P is a point on a rectangular hyperbola whose focus is S and directrix the axis of y Similarly for other points. For a rectangular hyperbola with its asymptotes on the axes of x and y, see § 22.

39. Take each horizontal tenth of an inch to represent £1 of capital, and each vertical *inch* to represent £1 of interest. Join the origin to the point (100, 3). The interest on £57 is shown by the ordinate corresponding to the abscissa 57. Interest = £1·7 = £1. 14s. Interest on £34 is represented by the ordinate whose abscissa is 34. Interest = £1·02 = £1 to the nearest shilling.

40. Take A for origin, AB 8 inches vertically to represent the chain AB. Take BD 6·4 inches horizontally to represent the weight of the chain. The tension at any point P varies as BP; for the tension at P = weight of chain below P. Mark the vertical inches 1, 2, 3, 4, etc.; and at these points draw horizontal distances ·1, ·4, ·9, 1·6, etc. Connecting by a curve the points thus obtained, we get the required graph. At 3 ft. 6 in. from the lower end the tension = $12\frac{1}{4}$ lbs. weight. At 6 ft. 3 in. from the lower end the tension = 39 lbs. nearly.

41. Mark the horizontal inches as seconds 0, 1, 2, etc, and the vertical ones as 0, 40 feet, 80 feet, etc. Join (0, 0) to

(10, 200). This represents the motion of the 1st particle. For the 2nd particle $s=4(t-2)^2$, since it does not start till after 2 seconds. When $t=3$, 4, 5, 6, 7, 8, 9, 10, $s=4$, 16, 36, 64, 100, 144, 196, 256. The graph is a curve through these points. At the intersection we have $t=8{\cdot}53$. By joining (7·5, 140) to (8·5, 160), and (8·5, 180) to (9·5, 200), and observing where these cut the curved graph, we get the times when they are 10 feet apart, viz. 8·23 secs., and 8·82 secs. At the end of the 4th second they are 64 feet apart.



42. Done in the text.

EXERCISES XLV.

1. $(2a)^2 = 4a^2$, also from a figure.

2. $(3a)^2 = 9a^2$, also from a figure.

3. Let CB be a side of the larger square. In CB mark off CD equal to a side of the smaller square. Produce BC to A making $CA = CB$. The rect. $AD \,.\, DB = CB^2 - CD^2$.

4. Let ABC be a △ right-angled at A. By II. 11 $AB^2 = BC^2 - AC^2 = (BC + AC)(BC - AC)$.

5. Let ABC be a △ right-angled at A. Let AD be perp. to BC. By II. 11. $BC^2 = AB^2 + AC^2 = AD^2 + BD^2 + AD^2 + DC^2 = BD^2 + DC^2 + 2AD^2$. By IV. 4. $BC^2 = BD^2 + CD^2 + 2BD \,.\, DC$ $\therefore$ $AD^2 = BD \,.\, DC$.

6. Let ABC be a △ in which $AB = AC$. Let D be any pt. of BC, E the mid. pt. of BC. $AB^2 - AD^2 = BE^2 + AE^2 - (DE^2 + AE^2) = BE^2 - DE^2 = BD \,.\, DC$ (II. 5).

7. Proved in the course of the proof of II. 11.; or thus, $AB^2 = BD^2 + AD^2 = BD^2 + BD \,.\, DC$ (Question 5) $= BD \,.\, BC$ (IV. 3.).

8. Let AB be divided equally at C, unequally at D. $AD^2 - DB^2 = (AD - DB)(AD + DB) = (AC + CD - \overline{AC - CD})\ AB = 2CD \,.\, AB$.

9. Let AB be the greater; in it cut off BC equal to the less. By IV. 7. $AB^2 + BC^2 = 2AB \,.\, BC + AC^2$, *i.e.* $AB^2 + BC^2$ is not less than $2AB \,.\, BC$.

10. Let ABC be the △ right-angled at A, D any pt. in BC, E the mid. pt. $\angle B = 45° = \angle BAE$ $\therefore$ $AE = BE$. By II. 8. $BD^2 + DC^2 = 2BE^2 + 2DE^2 = 2AE^2 + 2DE^2 = 2AD^2$.

11. Let PR, QS intersect at T. The diagonals of a rhombus bisect each other at rt. ∠s $\therefore$ PR bisects QS at rt. ∠s $\therefore$ RTP passes through O, since $OS = OQ$. $OP \,.\, OR = OT^2 - PT^2$ (IV. 6.) $= OT^2 + TS^2 - PT^2 - TS^2 = OS^2 - SP^2$.

12. If x, y be the length sides of the rect, a that of the square, $xy = a^2$ (hyp.). The perimeter of the rectangle $= 2(x+y)$, that of the square $= 4a$. $(x+y)^2 = (x-y)^2 + 4xy = (x-y)^2 + 4a^2$ $\therefore$ $(x+y)^2 > 4a^2$ $\therefore$ $x+y > 2a$.

13. Let x, y be the lengths of the two parts, a of the whole line. $2x^2 + 2y^2 = (x+y)^2 + (x-y)^2 = a^2 + (x-y)^2$ $\therefore$ $x^2 + y^2$ is a minimum when $x - y = 0$.

14. Draw PE, PF perp. to AB, BC. $\angle EAP = 45° = \angle EPA$ $\therefore$ $AE = EP = BF$. Similarly $EB = FC$. $AP^2 - BP^2 = AE^2 - EB^2$ (II. 11.) $= BF^2 - FC^2 = BP^2 - PC^2$ $\therefore$ $AP^2 + PC^2 = 2BP^2 =$ sq. on the diagonal of the sq. on BP.

15. $AB^2 = AD^2 + BD^2 + 2AD \cdot DB$ (II. 4.) $= AD^2 + BD^2 + 2CD^2$ (hyp.) $= AC^2 + BC^2$ (II. 11.) $\therefore$ ACB is a rt. $\angle$.

16. $AC^2 - AE^2 = AD^2 + CD^2 - AD^2 - DE^2 = CD^2 - DE^2 = (CD + DE)(CD - DE) = 4DF \cdot CF$.

17. By IV. 9. $BE^2 + DE^2 = 2OE^2 + 2OD^2$. But $DE^2 = AD^2 = AO^2 + OD^2 = 2OD^2$ $\therefore$ $BE^2 = 2OE^2 =$ sq. on diagonal of sq. on OE.

EXERCISES XLVI.

1. $12^2 > 6^2 + 8^2$ $\therefore$ the $\angle$ opposite to the side 12 is obtuse.

2. $13^2 < 9^2 + 12^2$ $\therefore$ the $\angle$ opposite to 13 is acute; and this is the greatest $\angle$ (I. 10.) $\therefore$ the angles are all acute.

3. AC, BD are equal and bisect each other at E. $PA^2 + PC^2 = 2PE^2 + 2AE^2$ (IV. 12.) $= 2PE^2 + 2BE^2 = PB^2 + PD^2$ (IV. 12.).

4. $AB^2 + 2AC \cdot CE = AC^2 + BC^2$ (IV. 11.). $AC^2 + 2AB \cdot BF = AB^2 + BC^2$ (IV. 11.) $\therefore$ by adding and removing the common parts from both sides $2AC \cdot CE + 2AB \cdot BF = 2BC^2$.

5. Let AD, BE, CF be the medians. Produce AO to H making $OH = AO$. By Ex. xx. 1. OBHC is a parm. $\therefore$ $OD = \frac{1}{2}OH = \frac{1}{2}OA$. Similarly for OE, OF. $2OB^2 + 2OC^2 = 4BD^2 + 4OD^2$ (IV. 12.) $= BC^2 + OA^2$. $2OC^2 + 2OA^2 = CA^2 + OB^2$. $2OA^2 + 2OB^2 = AB^2 + OC^2$ $\therefore$ by addition $3(OA^2 + OB^2 + OC^2) = AB^2 + BC^2 + CA^2$.

6. Let $AB = 8$ in., $BC = 9$ in., $\angle ABC = 60°$. Draw AD perp. to BC. ABD is half the equilateral $\triangle$ on AB $\therefore$ $BD = 4$ in. and $AD = 4\sqrt{3}$ in. $\therefore$ $AC^2 = AD^2 + CD^2 = 48 + 25 = 73$. $AC = \sqrt{73} = 8{\cdot}54$ in.

7. Let $AB = 6$ in., $BC = 8$ in., $\angle ABC = 120°$. Draw AD perp. to BC produced. ABD is half the equilateral $\triangle$ on AB $\therefore$ $DB = \frac{AB}{2} = 3$ in. and $AD = 3\sqrt{3}$ $\therefore$ $AC^2 = AD^2 + DC^2 = 27 + 121 = 148$ and $AC = \sqrt{148} = 12{\cdot}17$ in.

8. Let $AB = 6$, $BC = 8$, and $AC = 10$ cms. Also let AD, BE, CF be the medians. $\angle ABC =$ a rt. $\angle$ (II. 12.) $\therefore$ $BE = \frac{AC}{2} = 5$ cms. $AD^2 = AB^2 + BD^2 = 36 + 16 = 52$ $\therefore$ $AD = 7{\cdot}2$ cms. $CF^2 = CB^2 + BF^2 = 64 + 9 = 73$ $\therefore$ $CF = 8{\cdot}5$ cms.

9. Using IV. 12. the lengths of the medians are found to be $9{\cdot}8$, $9{\cdot}2$, and 6 cms.

10. Let AD be perp. to BC, $BD = \frac{3}{5}BC = 6$ cms., $CD = \frac{2}{5}BC = 4$ cms. $\therefore$ $AD^2 = AB^2 - BD^2 = 64 - 36 = 28$ $\therefore$ $AC^2 = AD^2 + CD^2 = 28 + 16 = 44$ and $AC = \sqrt{44} = 6{\cdot}63$ cms.

11. Let AD, BE, CF be the medians, O their intersection. As in Question 5, $2OB^2 + 2OC^2 = BC^2 + OA^2$. But $OA = \frac{2}{3}x$, $OB = \frac{2}{3}y$, $OC = \frac{2}{3}z$ $\therefore$ $\frac{8}{9}y^2 + \frac{8}{9}z^2 = BC^2 + \frac{4}{9}x^2$.

12. $9BA^2 = 8x^2 + 8y^2 - 4z^2 = 200 + 392 - 16 = 576$ $\therefore$ $BA^2 = 64$ $\therefore$ $BA = 8$.

13. $AC^2 = BC^2 + BA^2 - 2BD \cdot BC$ (IV. 11.), $\angle B$ being acute. $AC^2 = BC^2 + BA^2 - 2\,BF \cdot BA$ (IV. 11.), $\angle B$ being acute $\therefore$ $BD \cdot BC = BF \cdot BA$. IV. 10. would be used if B were obtuse.

14. Let A, B be the fixed pts., C the mid. pt. of AB, P the moving pt. $2CP^2 + 2AC^2 = AP^2 + BP^2$ (IV. 12.) = a constant (hyp.). AC is constant $\therefore$ CP is constant in length $\therefore$ the locus is a circle whose centre is C.

15. Let O be the mid. pt. of BC. Then $AB^2 + AC^2 = 2OB^2 + 2OA^2$ (IV. 12.) = a constant.

16. Let AD, BE, CF be the medians. $4AD^2 + 4BD^2 = 2AB^2 + 2CA^2$ $\therefore$ $4AD^2 = 2AB^2 + 2CA^2 - BC^2$. $4BE^2 = 2BC^2 + 2AB^2 - CA^2$. $4CF^2 = 2CA^2 + 2BC^2 - AB^2$ $\therefore$ $4(AD^2 + BE^2 + CF^2) = 3(AB^2 + BC^2 + CA^2)$.

17. Let ABCD be a parm. Then AC, BD bisect each other at E. $AB^2 + BC^2 + DC^2 + DA^2 = 2AB^2 + 2BC^2 = 4AE^2 + 4BE^2$ (IV. 12.) $= AC^2 + BD^2$

18. $PA^2+PC^2=2AO^2+2OP^2$. $PB^2+PD^2=2BO^2+2OP^2$ $\therefore$ $PA^2+PB^2+PC^2+PD^2=2AO^2+2BO^2+4OP^2=$ a constant.

19. Let ABC be an isosceles $\triangle$ on base BC, PBC another $\triangle$ such that AP is $\parallel$ to BC. The perp. AD bisects BC. $AB^2+AC^2=2BD^2+2AD^2$. $PB^2+PC^2=2BD^2+2PD^2$ (IV. 12.). But $AD<PD$ (I. 11.) $\therefore$ $AB^2+AC^2<PB^2+PC^2$.

20. Let ABCD be a quadl., E, F the mid. pts. of AC, BD. $(AB^2+BC^2)+(CD^2+DA^2)=2AE^2+2BE^2+2AE^2+2DE^2$ (IV. 12.) $=4AE^2+2BE^2+2DE^2=4AE^2+4BF^2+4EF^2$ (EF being the median of $\triangle BED$) $=AC^2+BD^2+4EF^2$.

21. Let $AB=7$, $BC=5$, $CA=8$ in. Draw BD perp. to AC. $AB^2=AC^2+BC^2-2\,.\,CD\,.\,CA$ (IV. 11.), *i.e.* $49=64+25-16CD$ $\therefore$ $CD=\frac{5}{2}$ in. $=\frac{1}{2}CB$ $\therefore$ $\angle ACB=60°$, for $\triangle DCB$ is half an equilateral $\triangle$.

22. Let O be the intersection of BD, AC. $AB^2-AE^2=BO^2+AO^2-EO^2-AO^2=BO^2-EO^2=BE\,.\,ED$ (IV. 5.). If E divides BD externally, use IV. 6.

23. $2HM^2+2AM^2-2KL^2-2AL^2=HK^2+AH^2-HK^2-AK^2$ (IV. 12.) $\therefore$ $2(HM^2-KL^2)=AH^2-AK^2+\frac{1}{2}AH^2-\frac{1}{2}AK^2$ $\therefore$ $8(HM^2-KL^2)=6(AH^2-AK^2)=3(2AH^2+2DH^2-2AK^2-2DK^2)=3(AB^2+AD^2-AC^2-AD^2)$ (IV. 12.) $=3(AB^2-AC^2)$.

24. $AB^2=AC^2+BC^2-2AC\,.\,CF$ (IV. 11.) $\therefore$ $BC^2=2AC\,.\,CF=$ twice figure ECFG.

25. Let ABCD be a quadl., AD and BC subtending obtuse $\angle$s at E the intersection of diagonals. $AD^2>AE^2+ED^2$, $BC^2>BE^2+EC^2$ (IV. 11.) $\therefore$ $AD^2+BC^2>AE^2+EB^2+EC^2+ED^2$. Similarly $AC^2+BD^2<AE^2+EB^2+EC^2+ED^2$.

26. $\angle AOB$ is gr. than $\angle D$ (I. 8.) $\therefore$ $\angle AOB$ is obtuse $\therefore$ $AB^2>AO^2+BO^2$. Similarly $BC^2>BO^2+CO^2$ and $CA^2>CO^2+OA^2$ $\therefore$ $AB^2+BC^2+CA^2>2(AO^2+BO^2+CO^2)$.

27. $4BE^2+4AE^2-4CF^2-4AF^2=2AB^2+2BC^2-2AC^2-2BC^2$ (IV. 12.) $\therefore$ $4BE^2-4CF^2+AC^2-AB^2=2AB^2-2AC^2$ $\therefore$ $4(BE^2-CF^2)=3(AB^2=AC^2)$.

28. Let AB be the given base. Since the area is given, the altitude is given. Draw AD perp. to AB and equal to the given altitude. Let APB be such a $\triangle$. Then $AP^2+BP^2=\frac{1}{2}AB^2+2EP^2$ (E being the mid. pt. of AB). Now AP^2+BP^2 is given and AB

is given $\therefore$ EP is known. Describe a circle with centre E and radius equal to this known value of EP. The point where this cuts the parallel to AB drawn through D is the required vertex.

EXERCISES XLVII.

1. DO . CO = AO . BO (IV. 13.), *i.e.* DO $= \frac{15}{7} = 2\frac{1}{7}$ in.

2. If r = radius, AO . OB + OD$^2 = r^2$ $\therefore$ $r^2 = 21 + 4 = 25$, and $r = 5$ in. We see that AB is a diameter, *i.e.* D lies at the mid. pt. of AB.

3. Let CO $= x$ in. so that DO $= 9 - x$ in. AO . OB = CO . OD (IV. 13.) $\therefore$ $20 = x(9 - x)$, whence $x = 5$ or 4 in., and DO = 4 or 5 in

4. OA . OB = OC . OD (IV. 14. Cor.) $\therefore$ OB $= \frac{7 \times 10}{5} = 14$ in. $\therefore$ AB = 9 in.

5. If OC is the tangent, OC^2 = OA . OB (IV. 14.) $= 4 \times 9 = 36$ $\therefore$ OC = 6 in.

6. If OC is a tangent, OC^2 = OA . OB (IV. 14.) = 36 $\therefore$ OC = 6 cm. $\therefore$ CD^2 = OD^2 − OC^2 $= 64 - 36 = 28$. CD $= \sqrt{28} =$ 5·29 cm.

7. On AB, on the same side as the pt. C, describe an equilateral $\triangle$AEB. E is the centre reqd. By measurement, the intercept on OC = 7·15 cm.

8. If DE is perp. to AB, DE^2 = OD^2 − OE^2 $= 8^2 - 7^2 = 15$ $\therefore$ DE = 3·87 cm.

9. Let O be the centre of the wheel, OC the vertl. rad., AB the face of the brick, so that CA = 12, and AB = 4 in. Produce AB to meet the circumference again at D. Draw OE perp. to BD. AB . AD = AC^2 (IV. 14.) $\therefore$ AD $= \frac{144}{4} = 36$ $\therefore$ DE $= \frac{1}{2}$BD = 16, OC = AE $= 4 + 16 = 20$.

10. Let BA be the common chord, meeting in E the common tangent CD. CE^2 = BE . EA = ED^2 (IV. 14.).

11. Let C be any point in the common chord BA produced, CD, CE tangents to the two circles. CD^2 = BC . CA = CE^2 (IV. 14.).

12. Let AB, AC be tangents. Draw ADE cutting the circle at D, E. AB^2 = AD . AE = AC^2 (IV. 14.).

13. Sq. on tangent CP = AC . CB = a constant, since A, B, C are fixed points ∴ locus of P is a circle whose centre is C.

14. Let AP be such a line drawn from a fixed pt. A. Let Q be its mid. pt, C the centre of circle. Bisect AC at D. DQ = $\frac{1}{2}$PC (Ex. xx. 1.) = a constant ∴ the locus of Q is a circle with centre at the fixed pt. D.

15. Proved in the last line but four of II. 11. Or thus: ∠DBO = 90° − ∠DOB = ∠BAD ∴ BO touches the circumcircle of △BDA ∴ OD . OA = OB^2 (IV. 14.).

16. Let AB cut CD at rt. ∠s in E. The centre O lies in AB (III. 3. Cor.). AE . EB = AO^2 − OE^2 (IV. 5.) = CO^2 − OE^2 = CE^2 = CE . ED.

17. CD . CE = CB . CA (IV. 14. Cor.), *i.e.* $x(x+c) = b(a+b)$.

18. Let D be the mid. pt. of BC. BE . BA = BD^2 = $\frac{1}{4}$BC^2 = $\frac{1}{4}$BA^2 ∴ BE = $\frac{1}{4}$BA ∴ EA = $\frac{3}{4}$BA.

19. AP . AQ = AC . AB ∴ PCBQ is a cyclic quadl. ∴ ∠CPQ + ∠CBQ = 180° (IV. 14.). But CPQ = 90° (IV. 18.) ∴ ∠CBQ = 90° ∴ Q lies on a str. line through B perp. to AB.

20. Let E, F be the mid. pts. of CD, AB. EF bisects AB at rt. ∠s (II. 1. 2.) ∴ EF contains the centre of the circle ∴ DE is a tangent ∴ DA . DO = DE^2 = $\frac{1}{4}$DA^2 ∴ DO = $\frac{1}{4}$DA.

21. Let AEB, CED be chords. AB^2 − CD^2 = (AE + EB)^2 − (CE + ED)^2 = (AE − EB)^2 + 4AE . EB − (CE − ED)^2 − 4CE . ED = (AE − EB)^2 − (CE − ED)^2 (IV. 13.).

22. The circle whose diameter is AB passes through P, Q, since the ∠s P, Q are rt. ∠s ∴ AO . OP = BO . OQ (IV. 13. or 14.).

23. Let AB, CD be the ∥ chords cut by PQ in R, S Let T be the mid. pt. of PQ. PT^2 − TR^2 = PR . RQ (IV. 5.) = AR . RB (IV. 13.) = CS . SD (hyp.) = QS . SP (IV. 13.) = QT^2 − TS^2. But PT = QT ∴ TR = TS ∴ the mid. pt. of PQ is the mid. pt. of RS and therefore lies on the line ∥ to AB and CD and equidistant from them.

24. ∠AEB = ∠EAB (I. 5.) = $\frac{1}{2}$∠EBC = $\frac{1}{2}$∠ECB (I. 5.) = ∠EDC ∴ AE touches the circle through E, B, D (III. 18.).

25. Proved on page 271.

EXERCISES XLVIII.

1. $AB^2 + BH^2 = 2AB \cdot BH + AH^2$ (IV. 7.) $= 2AH^2 + AH^2$ (hyp.).

2. $\triangle BAF = \triangle DAH$ in all respects (I. 4.) $\therefore \angle LBH = \angle ADH = 90° - \angle AHD = 90° - \angle LHB$ (I. 3.) $\therefore$ HLB is a rt. $\angle$.

3. $\triangle ACK = \frac{1}{2}$ fig. HC (II. 9.) $= \frac{1}{2}$ fig. FH (IV. 16.) $= \triangle AFK$ (II. 9.) $\therefore$ FC is $\parallel$ to AK (II. 7.). $\triangle AGK = \frac{1}{2}$ fig. FK (II. 9.) $= \frac{1}{2}$ fig. AC (IV. 16.) $= \triangle ABK$ (II. 9.) $\therefore$ GB is $\parallel$ to AK (II. 7.).

4. $\angle EOD = \angle HOB$ (I. 3.) $= 90° - EBL$ (Question 2) $= 90° - EFB$ (I. 5.) $= \angle FDO$ $\therefore$ $EO = ED$ (I. 6.) $= EA$ $\therefore$ $\angle AOD$ is a rt. $\angle$ (III. 18.).

5. Let AB be divided at H. $AH \cdot HB = AB \cdot BH - BH^2 = AH^2 - BH^2 = (AH + BH)(AH - BH)$.

6. Let $AH = x$, then $HB = 3 - x$ $\therefore$ $x^2 = 3(3 - x)$ $\therefore$ $x^2 + 3x = 9$ $\therefore$ $x = \frac{\sqrt{45} - 3}{2} = \frac{3}{2}(\sqrt{5} - 1) = 1{\cdot}85$. $3 - x = 1{\cdot}15$.

7. Let $AH = x$, then $BH = x - 3$ $\therefore$ $x(x - 3) = 9$ $\therefore$ $x = \frac{3}{2}(\sqrt{5} + 1) = \frac{3}{2} \times 3{\cdot}236 = 4{\cdot}85$ $\therefore$ $x - 3 = 1{\cdot}85$.

8. $AB \cdot AL = AH^2$ (hyp.) $\therefore$ $AL^2 + AL \cdot LB = AH^2$ $\therefore$ $AL^2 = AH^2 - AL \cdot LB = AH^2 - AL \cdot AH = AH \cdot LH$.

EXERCISES XLIX.

1. Let ABC be a rt. $\angle$. Construct a triangle ABD having the $\angle$s B, D each double of $\angle A$. Bisect $\angle ABD$ by BE. Bisect $\angle$s ABE, EBD. $\angle ABD = \frac{4}{5}$ of a rt. $\angle$ $\therefore$ $\angle DBC = \frac{1}{5}$ of a rt. $\angle$

2. $\angle BCD = \angle B = \frac{4}{5}$ of rt. $\angle$. $\angle ACD =$ supplement of $\angle BCD = \frac{6}{5}$ of a rt. $\angle = 3\angle A$.

3. AC and CD each subtend $\frac{2}{5}$ of a rt. $\angle$ at circumference, or $\frac{1}{5}$ of 4 rt. $\angle$s at the centre of the smaller circle $\therefore$ they are sides of an inscribed regular pentagon.

4. $\angle AED =$ supplement of ACD (III. 13.) $= \angle BCD = \frac{4}{5}$ of a rt. $\angle$. $\angle ADE = \angle AED$ (I. 5.) $= \frac{4}{5}$ of a rt. $\angle$ $\therefore$ $\angle DAE = \frac{2}{5}$ of a rt. $\angle$.

5. $\triangle ABD = \triangle ADE$ in all respects by the last example $\therefore$ circumcircle of $\triangle ABD =$ circumcircle of $\triangle ADE =$ circumcircle of $\triangle ACD$.

6. $\angle DAE = 36°$ (Example 4) $= \angle CDA$ $\therefore$ CD is $\parallel$ to AEK. $\angle BDC = 36° = \angle DAE = \angle DCE$ (III. 12.) $\therefore$ BDK is $\parallel$ to CE.

7. In a $\triangle$ having each of the base-angles double of the vertical $\angle$ divide the vertical angle into 4 equal parts. Each of these is $\frac{1}{10}$ of a rt. $\angle$.

8. The $\angle$ at the centre $= 2\angle CBD$ (III. 11.) $= 144° = 180° - A$ $\therefore$ the centre lies on the arc CD. Also the centre lies on the str. line bisecting CD at rt. $\angle$s $\therefore$ it lies at the mid. pt. of arc CD.

9. Draw a $\triangle ABC$ in which $\angle B = \angle C = 2\angle A$. Draw an equilat. $\triangle ABD$ on the same side of AB. $\angle DBC = (\frac{4}{5} - \frac{2}{3})$ of a rt. $\angle = \frac{2}{15}$ of a rt. angle. Half this angle is the one required.

10. $\angle BAE = \angle BAD + \angle DAE = (\frac{2}{3} + \frac{2}{5})$ of a rt. $\angle$ $\therefore$ BE subtends at the centre $\frac{1}{5}$ of 4 rt. $\angle$s $\therefore$ BE is the side of a regular inscribed pentagon.

11. BD subtends $\frac{2}{5}$ of a rt. $\angle$ at the centre $\therefore$ BD is the side of a regular inscribed decagon.

12. In the figure of IV. 17. draw DF perp. to BC. DF bisects BC, since DBC is isosceles. Let $AC = x$, $BC = a - x$; then $BD = x$. $x = \frac{\sqrt{5}-1}{2} . a$, $a - x = a - \frac{\sqrt{5}-1}{2} . a = \frac{3-\sqrt{5}}{2} . a$. $BF = \frac{1}{2}BC = \frac{3-\sqrt{5}}{4} . a$. $AF = a - BF = \frac{\sqrt{5}+1}{4} . a$ $\therefore$ $AF . FB = \frac{(\sqrt{5}+1)(3-\sqrt{5})}{16} a^2 = \frac{\sqrt{5}-1}{8} a^2 = \frac{1}{4} ax$.

13. Let A be the centre, BC a side of the inscribed decagon. ABC is a $\triangle$ with each base-angle double of $\angle A$ $\therefore$ $BC = \frac{r}{2}(\sqrt{5} - 1)$ ... page 255.

EXERCISES L.

2. The bisectors are concurrent (Example 1) $\therefore$ they are equal (I. 6.).

4. Draw two perpendicular diameters and join their ends. The sides are equal (I. 4.). Any angle is a rt. $\angle$ (III. 17.).

5. Draw two perp. diameters and draw tangents at their ends. The sides of the quadrilateral formed by these tangents are all equal; for each = a diameter (II. 2.). The angles are rt. $\angle$s; for each = an angle at the centre (II. 2.).

6. Let ABCD be the square, E, F, G, H the mid. pts. of AB, BC, CD, DA. Let EG, FH intersect at O. The figures formed

are rectangular parms. (II. 1.) ∴ EO, FO, GO, HO are all equal, since each is equal to half a side of the square. The circle whose centre is O and radius EO is the one required (III. 5.).

7. The diagonals of a square are equal, and bisect each other ∴ the circle whose centre is their intersection and radius half a diagonal is the one required.

8. By drawing radii OA, OB, OC, OD, OE including ∠s of 72° we obtain 5 equal arcs. Draw tangents LAF, FBG, GCH, HDK, KEL. The quadrilateral AFBO is divided by FO into two △s equal in all respects (I. 17.) ∴ ∠FOB = 36°. Similarly ∠GOB = 36° ∴ △s FBO, GBO are equal in all respects ∴ FG = 2FB = 2FA = FL. Similarly all the sides of the pentagon are equal. Also each ∠ = supplement of ∠ at centre = 108° ∴ the pentagon is regular.

9. Bisect the angles. The bisectors are concurrent and equal (Ex. l. 1 2.) ∴ the perps. from the point of concurrence to the sides are equal (I. 16.). With any one of these perps. as radius the circle may be described.

10. Bisect the angles. The bisectors are concurrent and equal (Ex. l 1. 2.). With any one of these as radius the circle may be described.

11. See Ex. xxxviii. 22.

12. The vertical ∠ of a △ whose base-angles are each twice the vertical ∠ = 36°. The angle of an equilat. △ = 60°. The difference = 24°. Place an angle of 24° at the centre The chord subtended is one side of the regular quindecagon. Place equal chords consecutively in the circle, and the required figure is described. The figure is equilateral by construction. It is also equiangular: for each ∠ subtends $\frac{13}{15}$ of the circumference.

13. EF may be cut off on either side of E ∴ there are 2 solutions.

16. Let A be the given pt., BC the given str. line which is to contain the centre. Draw AD perp. to BC, produce AD to E making DE = AD. By III. 1. the circle must pass through E as well as A ∴ the problem is the same as the one in Question 13.

17. On OP as diameter describe a circle. Place in it a chord OC equal to a side of the given square. Describe a circle with centre P and radius PC cutting the given line in A, B. $OA . OB = OC^2$ (IV. 14.).

19. Draw a str. line OAB making OA, OB equal to the sides of the given rectangle. Describe any circle through A, B. Draw a tangent OC. OC is a side of the required square (IV. 14.).

20. Let AB be a side of the given square. Draw a circle touching AB at B. With centre A and radius AC equal to the given side of the rectangle cut the circle at C. Let AC cut the first circle at D. The rect. contained by AC, $AD = AB^2$ and is therefore the required rectangle.

21. Draw any circle through the given pts. A, B cutting the given circle at C, D. Let AB, CD meet at E. Through E draw a diameter EFG of the given circle. The circle described through ABF must pass through G (IV. 14. Cor.).

22. The intersection of the common chords is the point. Prove by IV. 14.

23. Describe a circle about ABC Draw the tangent at A meeting BC produced at D. $AD^2 = BD . DC$ (IV. 14.).

24. Let h be the height of flagstaff AB, k that of the tower BC. Let D be the point of contact of the horizontal through C and a circle through A, B. Let E be any other pt. in CD. Join AE cutting the circle at F. $\angle ADB = \angle AFB$ (III.12.) $> \angle AEB$ (I. 8.). Thus D is the required point; and $CD^2 = BC . CA$ (IV. 14.) $\therefore$ $CD = \sqrt{k(k+h)}$.

25. Let AB be the given str. line. On AB as diameter describe a circle. Take centre C, and at any pt. P draw a tangent PQ equal to a side of the given square. Produce AB to R, making $CR = CQ$. Draw a tangent RS. $AR . RB = RS^2$ (IV. 14.) $= PQ^2$ (I. 17) = the given square.

26. Let AB be the given str. line. On AB describe a semicircle ADB. Draw a str line $\parallel$ to AB at a distance equal to a side of the given square, and let one of the points of intersection with the semicircle be D. Draw DE perp. to AB. $AE . EB = DE^2$ (IV. 13.) $\therefore$ E is the required point. If the side of the given square is gr. than $\frac{1}{2}$AB, the problem is impossible.

27. Let AB be a diamr. of the circle. Make $\triangle$ADC having its sides 3, 4, and 5 cms. long, the 5 cm. side AC lying along AB. Produce AD to meet the circle at E. AEB is the $\triangle$ reqd. (I. 19. 22.).

28. With centre B and radius equal to a side of the given square describe a circle. Draw from A a tangent AP. In AB cut off AC equal to AP. $AB^2 - AC^2 = AB^2 - AP^2 = BP^2 =$ the given square.

29. Let O be the centre. Draw OD a radius perp. to OA, OE, OF perp. to OB, OC. The $\triangle$DEF is the $\triangle$ABC turned through 90° without any alteration of size or shape.

30. Bisect AB at O. Draw OE perp. to AB. Make $\angle$s CAF, ACF each 45°. With centre A and radius AF describe a circle cutting OE at E. In OB cut off OD equal to OE. $AD^2 + DB^2 = 2AO^2 + 2OD^2$ (IV. 8.) $= 2AO^2 + 2OE^2 = 2AE^2$ (II. 11.) $= AF^2 + FC^2 = AC^2$. $2AC^2 = 2AD^2 + 2DB^2 = (AD + DB)^2 + (AD - DB)^2$ $\therefore$ $2AC^2$ is a minimum when $AD = DB$ $\therefore$ the least value of $2AC^2$ is $4AD^2$ when $AD = \frac{1}{2}AB$, *i.e.* $2AC^2$ must not be less than AB^2.

31. Let AB be the str. line bisected at C. Take D any pt. in AB. $AD \cdot DB = AC^2 - CD^2$ $\therefore$ $AD \cdot DB$ is a maximum when CD is a minimum, *i.e.* when D is at C.

32. Let p be length of perp. Then $p \times 26 =$ twice area $=$ side $24 \times$ perp. $= 240$ $\therefore$ $p = \frac{120}{13} = 9\frac{3}{13}$.

EXERCISES LI.

1. Let A be a pt. of intersection, AC, AD tangents. Since AD touches one circle and DAC is a rt. $\angle$, AC passes through the centre of this circle: but AC is a tangent to the other circle $\therefore$ a tangent to one circle passes through the centre of the other.

2. Let A be the given point of intersection. The centres of all the circles must lie on the tangent at A to the given circle (Question 1).

3. Let A be the given pt, B the given pt. of intersection with the circle. Draw BC touching the given circle. Make an $\angle$BAC equal to $\angle$ABC. AC = CB (I. 6.) $\therefore$ the circle with centre C and radius CA passes through A and cuts the given

circle orthogonally at B, since CB is a tangent to one circle and radius of the other.

4. The centres of the circumcircles are at F, E, the mid. pts. of AB, AC. $\angle$FDA $=\angle$FAD (I. 5.) and $\angle$EDA $=\angle$EAD $\therefore$ whole $\angle$FDE $=\angle$FAE $= 90°$, *i.e.* the radii are at rt. $\angle$s $\therefore$ the tangents at D are at rt. $\angle$s.

5. Let the circle whose radius is PA cut at D a circle through B, C. PB . PC $=$ PA^2 (IV. 14.) $=$ PD^2 $\therefore$ PD touches the circle DBC. But PD is a radius of the circle DA $\therefore$ the circles cut orthogonally.

6. Let P, Q be the points of contact. Let a circle through T, U meet the first circle at R. CR^2 $=$ CP^2 $=$ CU . CT (Ex. xlv. 7.) $\therefore$ CR touches the circle TUR $\therefore$ the circles cut orthogonally.

EXERCISES LII.

1. Let PQ be a common tangent meeting the radical axis in T. TP $=$ TQ (property of radical axis).

2. Let AB, AC be tangents drawn from a pt. A on the radical axis. Let D, E be the centres. AC $=$ AB since A is on the radical axis $\therefore$ the circle with centre A and radius AB goes through C. Also it is touched by BD, CE, since the $\angle$s B, C are rt. $\angle$s; $\therefore$ it cuts both circles orthogonally.

3. Draw tangents from the radical centre, and use any of these as radius.

4. Let A, B, C be the pts. of contact, E, F the centres of the circles which touch at A; D the other centre. Let the tangents at A, B meet at T. T lies on the bisector of $\angle$AFB (from $\triangle$TAF, TBF). Similarly any other pair of tangents meet on the bisector of an $\angle$ of the $\triangle$DEF. But the bisectors are concurrent $\therefore$ the tangents are concurrent.

Nos. 5 and 6 are particular cases of No. 3. In No. 5 two of the circles are of infinitely small radius; in No. 6 one circle is so.

EXERCISES LIII.

1. Let the diagonals of the quadl. ABCD meet at O. $\frac{\triangle AOD}{\triangle AOB} = \frac{DO}{BO} = \frac{\triangle COD}{\triangle COB}$ (V. 1.).

2. Let ABCD be a trapezium having AB $\parallel$ to CD. Let the diagonals AC, BD meet at O. AB is $\parallel$ to the base CD of $\triangle$DOC $\therefore \frac{CO}{OA} = \frac{DO}{OB}$ (V. 2. Cor.).

3. From D a pt. in the base BC let DE, DF $\parallel$ to AB and AC respectively meet AC at E and AB at F. Let AD, EF meet at O. AO = OD (II. 2.) $\therefore$ the locus of O is a str. line $\parallel$ to BC and bisecting the sides AB, AC (V. 2.).

4. Let DEF be the mid. pts. of the sides BC, CA, AB of $\triangle$ABC. Join BE, CF. $\triangle BFE = \triangle AFE$ (II. 6.) $= \triangle EFC$ (II. 6.) $\therefore$ EF is $\parallel$ to BC (II. 7.). Similarly DF is $\parallel$ to CA $\therefore$ DFEC is a parm. $\therefore$ EF = CD = $\frac{1}{2}$BC. Similarly DE = $\frac{1}{2}$AB, and DF = $\frac{1}{2}$AC.

5. Let the three $\parallel$ str. lines AB, CD, EF cut off intercepts AC, CE, BD, DF on the str. lines ACE, BDF. Join E, B cutting CD at G. $\frac{AC}{CE} = \frac{BG}{GE}$ (V. 2) $= \frac{BD}{DF}$ (V. 2.).

6. $\frac{BG}{BD} = \frac{BE}{BA}$ (V. 2.) $= \frac{BF}{BC}$ (V. 2.) $\therefore$ GF is $\parallel$ to CD (V. 2.).

7. Draw PD $\parallel$ to AB to meet BC at D. Produce BD to C making DC = BD. Join CP and produce it to meet BA at A. $\frac{CP}{PA} = \frac{CD}{DB}$ (V. 2.) $\therefore$ CP = PA.

8. Draw DE $\parallel$ to BC to meet AC at E. In AC produced take CF = CE. Join DF meeting BC at H. $\frac{DH}{HF} = \frac{EC}{CF}$ (V. 2.) $\therefore$ DH = HF $\therefore$ DHF is the line reqd.

9. $\triangle DBE = \triangle DCE$ (II. 5.) $\therefore \triangle DBF = \triangle ECF$. Also $\frac{\triangle ADF}{\triangle BDF} = \frac{AD}{DB}$ (V. 1.) $= \frac{AE}{EC}$ (V. 2.) $= \frac{\triangle AEF}{\triangle ECF}$ (V. 1.) $\therefore \triangle ADF = \triangle AEF$.

10. Join AO. $\triangle ANO = \frac{1}{2}$ parm. $ANOM = \triangle AMO = \triangle OMN$ (II. 2.) $\therefore \frac{\triangle BNO}{\triangle OMN} = \frac{\triangle BNO}{\triangle AON} = \frac{BN}{AN}$ (V. 1.) $= \frac{BO}{OC}$ (V. 2.) $= \frac{AM}{CM}$ (V. 2.) $= \frac{\triangle AMO}{\triangle CMO}$ (V. 1.) $= \frac{\triangle OMN}{\triangle CMO}$.

11. $\frac{\triangle BED}{\triangle AED} = \frac{BE}{EA}$ (V. 1) $= \frac{1}{2} \therefore \triangle BED = \frac{1}{2}\triangle AED$. $\frac{\triangle CED}{\triangle AED} = \frac{CD}{DA}$ (V. 1.) $= \frac{2}{1} \therefore \triangle CED = 2\triangle AED \therefore \triangle CED = 4\triangle BED \therefore \frac{\triangle BED}{\triangle CED} = \frac{1}{4}$.

12. Let AH meet DE at M. Join BM cutting FG at O. $\frac{\triangle FAH}{\triangle ADH} = \frac{AF}{AD}$ (V. 1.) $= \frac{BH}{BD}$ (V. 2.) $= \frac{BO}{BM}$ (V. 2.) $= \frac{\triangle HOB}{\triangle HBM}$. Also $\triangle DAB = \triangle MAB$ (II. 5.) $\therefore \triangle ADH = \triangle HBM \therefore$ from the above $\triangle FAH = \triangle HOB$. And they are between the same parallels $\therefore FH = HO$. Thus if $HO = HF$, BO produced meets AH on DE. In the same way, since $KO = KG$, we can prove BO produced meets CK on DE $\therefore$ AH and CK meet on DE.

13. $\frac{FK}{DK} = \frac{AK}{CK}$ (V. 2.) $= \frac{CL}{AL} = \frac{GL}{DL}$ (V. 2.) $\therefore$ FG is $\parallel$ to AC (V. 2.).

14. AC is $\parallel$ to BD (V. 2.). Let BO be gr. than AO, so that DO is gr. than CO. $\frac{OB}{OA} = \frac{OD}{OC} \therefore \frac{OB - OA}{OA} = \frac{OD - OC}{OC}$, *i.e.* $\frac{2OP}{OA} = \frac{2OQ}{OC} \therefore$ PQ is $\parallel$ to AC and $\therefore$ also to BD (V. 2.).

15. With the figure of II. 10. $\frac{\text{parm. } KG}{\text{parm. } HG} = \frac{KE}{HE}$ (V. 1.) $= \frac{\text{parm. } FK}{\text{parm. } FH}$ (V. 1.) $= \frac{\text{parm. } HG}{\text{parm. } FH}$.

16. Let the medians AD, BE, of $\triangle ABC$ meet at O. Join OC. $\triangle BEC = \frac{1}{2}\triangle ABC = \triangle ADC \therefore \triangle BOD = \triangle AOE = \triangle COE$ (II. 6.). But $\triangle COD = \triangle BOD$ (II. 6.) $\therefore \triangle BOC = 2\triangle COE \therefore BO = 2OE$ (V. 1.). Similarly it may be shown that CF divides BE in the ratio of 2 to 1 $\therefore$ the medians are concurrent.

17. $\frac{\triangle AEB}{\triangle ADB}=\frac{AE}{AD}=\frac{1}{3}$ (V. 1.) $\therefore$ $\triangle AEB=\frac{1}{3}$ $\triangle ADB=\frac{1}{6}$ parm. ABCD (II. 2.).

18. $\frac{\triangle AED}{\triangle ABD}=\frac{DE}{DB}$ (V. 1.) $=\frac{3}{7}$ $\therefore$ $\triangle AED=\frac{3}{7}\triangle ABD=\frac{3}{14}$ parm. ABCD $\therefore$ $\triangle AED$: parm. ABCD : : 3 : 14.

EXERCISES LIV.

1. Let AD bisect $\angle BAC$ and base BC of the $\triangle ABC$. $\frac{BA}{AC}=\frac{BD}{DC}=1$ (V. 3.) $\therefore$ $BA=AC$.

2. $\frac{AE}{EB}=\frac{AD}{BD}$ (V. 3.) $=\frac{AD}{DC}=\frac{AF}{FC}$ (V. 3.) $\therefore$ EF is $\parallel$ to BC (V. 2.).

3. Let AD, BE, the bisectors of $\angle$s A and B of $\triangle ABC$ meet at G. $\frac{GA}{GD}=\frac{BA}{BD}$ from $\triangle ABD$ (V. 3.). Also since AD bisects $\angle A$ $\frac{BA}{AC}=\frac{BD}{DC}$ (V. 3.) $\therefore$ $\frac{BA}{BD}=\frac{AC}{DC}$ $\therefore$ $\frac{GA}{GD}=\frac{BA}{BD}=\frac{AC}{DC}=\frac{BA+AC}{BD+CD}=\frac{BA+AC}{BC}$. In the same way it may be shown that the bisector CF of $\angle C$ divides AD in the same ratio $\therefore$ AD, BE, CF are concurrent.

4. AO, BO, CO bisect the $\angle$s A, B, C. Hence, as in the preceding example, it may be proved that $\frac{AO}{OD}=\frac{BA+AC}{BC}$.

5. Let the bisectors of $\angle$s A and C meet BD at E. Also let the bisector of $\angle D$ meet AC at F. $\frac{DA}{AB}=\frac{DE}{EB}$ (V. 3.) $=\frac{DC}{CB}$ (V. 3.) $\therefore$ $\frac{DA}{DC}=\frac{AB}{CB}$. Also $\frac{AF}{CF}=\frac{AD}{CD}$ (V. 3.) $=\frac{AB}{CB}$ $\therefore$ BF bisects $\angle B$ (V. 3.) $\therefore$ the bisectors of $\angle$s B and D meet on AC.

6. AB bisects CD at rt. $\angle$s (III. 3.) $\therefore$ arc $BC=$ arc BD $\therefore$ $\angle BGC=\angle BGD$ $\therefore$ $\frac{CG}{GD}=\frac{CE}{ED}$ (V. 3.). Similarly $\frac{CF}{FD}=\frac{CE}{ED}$ $\therefore$ $\frac{CG}{GD}=\frac{CF}{FD}$.

7. Let OA and OB meet the inner circle at E and F, and let GOH be the common tangent at O to the circles, G lying on the same side of OC as the pt. A. $\angle$OFC = supplement of $\angle$OEC = $\angle$AEC. $\angle$OCF = $\angle$HOF (III. 18.) = $\angle$EAC (III. 18.) $\therefore$ $\angle$FOC = $\angle$ECA (I. 22.) = $\angle$COE (III. 18.) $\therefore \frac{AO}{OB} = \frac{AC}{BC}$ (V. 3.).

8. AD the altitude of the isos. $\triangle$ABC bisects the vertical $\angle$ at A $\therefore$ O the centre of the incircle lies in AD. Also BO and CO bisect the $\angle$s at B and C. From $\triangle$ABD $\frac{OD}{OA} = \frac{BD}{BA}$ (V. 3.)
$\therefore \frac{OD}{OA+OD} = \frac{BD}{BA+BD} \therefore \frac{OD}{AD} = \frac{2BD}{2(BA+BD)} = \frac{\text{base}}{\text{perimeter}}$.

9. Let AD meet the base at O. Produce BD to meet AC produced at F. Bisect DF at E and join OE From $\triangle$s BAD, FAD, BD = FD (I. 16.) $\therefore \frac{BE}{EF} = \frac{3}{1} = \frac{BA}{AC} = \frac{BO}{OC}$ (V. 3.) $\therefore$ OE is $\parallel$ to CF (V. 2.) $\therefore \frac{DO}{AO} = \frac{DE}{FE}$ (V. 2.) $\therefore$ DO = AO.

10. $AE = \frac{4}{7}AB$ and $AF = \frac{2}{7}AD$ $\therefore \frac{AE}{AF} = \frac{4}{2} = \frac{2}{1}$ and $\frac{EG}{FG} = \frac{AE}{AF}$ (V. 3.) $= \frac{2}{1}$.

11. Let the pt. O fall within BD. $\frac{BD}{CD} = \frac{BA}{AC}$ (V. 3.) $\therefore \frac{BD-CD}{BD+CD} = \frac{BA-AC}{BA+AC}$, *i.e.* $\frac{2OD}{2OB} = \frac{BA-AC}{BA+AC} \therefore \frac{OD}{OB} = \frac{BA-AC}{BA+AC}$.

EXERCISES LV.

1. Let AO bisect $\angle$A and BO bisect the extr. $\angle$ at B of $\triangle$ABC. Join CO, and let AO meet BC at F. $\frac{AB}{BF} = \frac{AO}{OF}$ (V. 4.). Also $\frac{AB}{AC} = \frac{BF}{CF}$ (V. 3.) $\therefore \frac{AB}{BF} = \frac{AC}{CF} \therefore \frac{AC}{CF} = \frac{AO}{OF} \therefore$ CO bisects the extr. $\angle$ at C, which proves the proposition.

2. Produce DP to E and AP to F. PA bisects ext. $\angle$ of $\triangle$CPD $\therefore \frac{AD}{AC} = \frac{PD}{PC}$ (V. 4.). $\angle$APB = a rt. $\angle$ (III. 17.) $\therefore$ $\angle$BPD = complement of $\angle$FPD = complement of $\angle$EPA = complement

of $\angle APC = \angle CPB$, *i.e.* PB bisects $\angle CPD$ $\therefore \frac{PD}{PC} = \frac{BD}{BC}$ (V. 3.) $\therefore \frac{AD}{AC} = \frac{BD}{BC}$ $\therefore \frac{AC}{BC} = \frac{AD}{BD}$.

3. Let the bisectors of the int. and ext. $\angle$s at P meet AB at C and D. $\frac{AC}{CB} = \frac{AP}{PB}$ (V. 3.) $\therefore$ C is a fixed pt. $\frac{AD}{DB} = \frac{AP}{PB}$ (V. 4.) $\therefore$ D is also a fixed pt. Also CPD is a rt. $\angle$ $\therefore$ the locus of P is a circle on CD as diameter.

4. $\angle OAB = \angle ODC$, $\angle BOA = \angle COD$ $\therefore \angle OBA = \angle OCD$ $\therefore \angle OBC = \angle OCB$ $\therefore$ OB = OC $\therefore$ AB = CD (I. 4.). OA bisects ext. $\angle$ of $\triangle BOD$ $\therefore \frac{AD}{AB} = \frac{OD}{OB} = \frac{OA}{OC} = \frac{AB}{BC}$ (V. 3.).

5. Let AD, the bisector of $\angle A$ of $\triangle BAC$, meet the base BC at D; and let AE, the bisector of the ext. $\angle$ at A, meet BC produced at E. $\angle DAE$ = a rt. $\angle$. $\frac{EB}{EC} = \frac{AB}{AC}$ (V. 4.) $= \frac{BD}{BC}$ $\therefore$ E is a fixed pt. $\therefore$ the locus of A is a circle on DE as diameter.

EXERCISES LVI.

1. Take D, E, F the mid. pts. of the sides BC, CA, AB of $\triangle ABC$. $\frac{AF}{FB} = 1 = \frac{AE}{EC}$ $\therefore$ EF is $\parallel$ to BC (V. 2.). Similarly DE is $\parallel$ to AB and DF to CA. Also $\triangle AEF$ is equiangular to $\triangle ABC$ (I. 20.) and $\therefore$ similar to it (V. 5.), and in the same way $\triangle$s CED, BFD are similar to $\triangle ABC$. Also FEDB is a parm. $\therefore \triangle FED = \triangle BDF$ in all respects (II. 2. and I. 4.). Similarly each of the $\triangle$s AFE, EDC is equal to $\triangle DEF$ in all respects $\therefore \triangle AEF = \triangle BDF = \triangle CDE = \triangle DEF$, and each is similar to $\triangle ABC$.

2. With the fig. of IV. 13. $\angle CEA = \angle BED$ (I. 3.), $\angle ECA = \angle EBD$ in the same segment $\therefore \triangle$s CEA, BED are equiangular, and $\therefore$ similar (V. 5.) $\therefore \frac{CE}{EA} = \frac{BE}{ED}$ $\therefore$ rect. CE . ED = rect. BE . EA.

3. With the fig. of IV. 14, $\angle OCA = \angle OBC$ in alternate segment, and $\angle COA$ is common to $\triangle$s OCA, OBC $\therefore$ these $\triangle$s are equiangular $\therefore \frac{OA}{OC} = \frac{OC}{OB}$ (V. 5.) $\therefore$ rect. $OA . OB = OC^2$.

4. $\angle$OBC $=\angle$ODA in the same segment, and $\angle$O is common to $\triangle$s BCO, DAO $\therefore$ these $\triangle$s are equiangular $\therefore \frac{OB}{OC}=\frac{OD}{OA}$ (V. 5.) $\therefore$ rect. OA . OB $=$ rect. OC . OD.

5. Let AD, BE two medians of $\triangle$ABC cut at G. $\frac{CE}{EA}=\frac{CD}{DB}$ $\therefore$ DE is $\|$ to AB (V. 2.) $\therefore$ $\triangle$CDE is equiangular to $\triangle$CBA, and $\therefore$ similar (V. 5.) $\therefore \frac{DE}{AB}=\frac{DC}{BC}=\frac{1}{2}$. Also $\triangle$s DGE, AGB are equiangular (I. 20.) and $\therefore$ similar (V. 5.) $\therefore \frac{DG}{AG}=\frac{DE}{AB}=\frac{1}{2}$, which proves the proposition.

6. Let AB be the man CD the post, AE the shadow of the man, so that DBE is a str. line. Let AE $=x$. $\triangle$s EAB, ECD are equiangular $\therefore \frac{EA}{AB}=\frac{EC}{CD}$, *i.e.* $\frac{x}{6}=\frac{x+4}{12}$ $\therefore x=4$ ft.

7. Let AB be the man, CD the post, AE the shadow of the man ($=10$ ft.). Let AC $=x$. $\triangle$s EAB, ECD are equiangular, $\therefore \frac{EA}{AB}=\frac{EC}{CD}$, *i.e.* $\frac{10}{6}=\frac{x+10}{12}$ $\therefore x=10$ ft.

8. Let AB be the pole, AC its shadow; DE the height of the house, DF its shadow. $\triangle$s CAB, FDE are equiangular, $\therefore \frac{DE}{DF}=\frac{BA}{AC}$, *i.e.* $\frac{DE}{60}=\frac{10}{20}$ $\therefore$ DE $=30$ ft.

9. Let $\triangle$ABC be such that BC $=5$, AB $=9$, AC $=7$ cms.; also let DEF be a similar $\triangle$ having EF $=3$ cms. $\frac{DE}{EF}=\frac{AB}{BC}$, *i e.* $\frac{DE}{3}=\frac{9}{5}$ $\therefore$ DE $=5{\cdot}4$ cms. $\frac{DF}{EF}=\frac{AC}{BC}$, *i.e.* $\frac{DF}{3}=\frac{7}{5}$ $\therefore$ DF $=4{\cdot}2$ cms.

10. Let AB $=x$, and AC $=y$ ft. As in lvi. 4, $\triangle$s OAC, ODB are similar $\therefore \frac{OB}{OD}=\frac{OC}{OA}$, *i.e.* $\frac{x+5}{12}=\frac{4}{5}$ $\therefore x=4{\cdot}6$ ft. Also $\frac{AC}{AO}=\frac{BD}{DO}$, *i.e.* $\frac{y}{5}=\frac{7}{12}$ $\therefore y=2$ ft. 11 in.

11. Let EF be the crease cutting AB at E and CD at F, and AC at O. When the folding is done, $\angle$EOA coincides with $\angle$EOC and is $\therefore$ equal to it. $\therefore$ $\angle$EOA $=\angle$EOC $=$ a rt. $\angle$. $\triangle$EAO

is similar to $\triangle CAB$ $\therefore \frac{EO}{AO}=\frac{BC}{BA}=\frac{6}{8}$ $\therefore$ $EO=\frac{3}{4}AO=\frac{3}{8}AC=\frac{3}{8}\sqrt{6^2+8^2}=\frac{15}{4}$ ft. $\therefore$ $EF=\frac{15}{2}=7\frac{1}{2}$ ft.

12. Let EOF be the crease, cutting AB at E, AD at O, AC at F. As in the previous example, EOF is perp. to AOD, and AO $=$ OD $\therefore$ EOF is $\parallel$ to BC, and since AO $=$ OD, AE $=$ EB, and AF $=$ CF $\therefore$ $EF=\frac{1}{2}BC$.

13. Let OE be the crease, cutting AB at O and AC at E. AO $=$ OB, and OE is perp. to AOB as in the previous examples $\therefore$ $\triangle$s AOE, ACB are equiangular $\therefore \frac{OE}{AO}=\frac{BC}{AC}$, *i.e.* $\frac{OE}{\frac{13}{2}}=\frac{5}{12}$ $\therefore$ OE $=\frac{65}{24}=2\cdot71$ in. nearly.

14. Let DE be the line $\parallel$ to BC cutting AB at D and AC at E. $\triangle$s ADE, ABC are similar $\therefore \frac{DE}{BC}=\frac{AD}{AB}$, *i.e.* $\frac{DE}{9}=\frac{4}{7}$ $\therefore$ DE $=5\frac{1}{7}$ in.

15. Let ABCD be a trapezium, having AB parallel to CD and equal to 2CD. Let the diagonals AC, DB meet at O. $\triangle$s DOC, BOA are equiangular (I. 20.) $\therefore \frac{AO}{OC}=\frac{BO}{OD}=\frac{AB}{CD}=2$, *i.e.* AO $=$ 2OC, and BO $=$ 2OD, which proves the proposition.

16. Let OAB, OCD, OEF intercept on the $\parallel$ str. lines ACE, BDF, portions AC, CE, and BD, DF. $\frac{AC}{BD}=\frac{OC}{OD}$ (for $\triangle$s OCA, ODB are equiangular) $=\frac{CE}{DF}$ (for $\triangle$s OCE, ODF are equiangular).

17. Let C, D be the centres of the circles, and AB a common tangent meet CD produced at O. In $\triangle$s OAC, OBD, $\angle$OBD $=$ a rt. $\angle=\angle$OAC $\angle$O is common $\therefore$ the $\triangle$s are equiangular $\therefore \frac{OC}{OD}=\frac{AC}{BD}$. Similarly if a common tangent EF divide CD internally at P, $\triangle$s CEP, DFP are equiangular $\therefore \frac{CP}{PD}=\frac{CE}{DF}$.

18. Let AD be the median to the base BC of $\triangle$ABC. Draw EF $\parallel$ to BC, to meet the median at G. $\frac{EG}{BD}=\frac{AG}{AD}$ for $\triangle$s AGE, ADB are equiangular (I. 20.) $=\frac{GF}{CD}$ for $\triangle$s AGF, ADC are equiangular $\therefore$ EG $=$ GF.

19. Let $\angle$ADB be obtuse, so that $\angle$ADC is acute. Let P be the centre of the circum-circle to $\triangle$ADB, and Q that of the circum-circle to $\triangle$ADC. P lies in PE which bisects AB at rt. $\angle$s. Q lies in QF which bisects AC at rt. $\angle$s. Also $\angle APB = 2$ supplement of $\angle$ADB (III. 11.) $= 2\angle ADC = \angle AQC$ (III. 11.) $\therefore \angle BPE = \angle AQF$. Also $\angle PEB =$ a rt. $\angle = \angle QFA$ $\therefore$ $\triangle$s PBE, QAF are equiangular $\therefore \dfrac{PB}{QA} = \dfrac{BE}{FA} = \dfrac{AB}{AC}$.

20. Let the perpendiculars AM, BN on a str. line DE be in a constant ratio. Let DE cut AB at O. $\triangle$s AMO, BNO are equiangular $\therefore \dfrac{AO}{BO} = \dfrac{AM}{BN} =$ a constant $\therefore$ O is a fixed pt., *i.e.* DE passes thro. a fixed pt. O in AB.

21. $\dfrac{BO}{OE} = \dfrac{AO}{OD}$ $\therefore$ DE is $\parallel$ to AB (V. 2.) $\therefore$ $\triangle$s AOB, DOE are equiangular (I. 20.) $\therefore \dfrac{AB}{DE} = \dfrac{BO}{OE} = \dfrac{2}{1}$. Also $\triangle$s CDE, CBA are equiangular (I. 20.) $\therefore \dfrac{CD}{CB} = \dfrac{CE}{CA} = \dfrac{DE}{AB} = \dfrac{1}{2}$ $\therefore$ CB and CA are bisected at D and E.

22. $\triangle$s QAO, QCD are equiangular (I. 20.) $\therefore \dfrac{QO}{QD} = \dfrac{AO}{CD}$ (V. 5.) $= \dfrac{BO}{CD} = \dfrac{PO}{PD}$ for $\triangle$s PBO, PCD are equiangular (I. 20) $\therefore \dfrac{QO}{OP} = \dfrac{QD}{PD}$.

23. Let DA produced meet EC at F. From $\triangle$s BAD, CAF, AD = AF (I. 20. and 16.). $\triangle$s BOE, AOD are equiangular (I. 20.) $\therefore \dfrac{BO}{OA} = \dfrac{BE}{DA}$ (V. 5.) $= \dfrac{DF}{DA}$ (for BEFD is a parm.) $= 2$ $\therefore$ BO = 2OA $\therefore$ O is a pt. of trisection of AB.

24. Let AB be $>$ AC, and let CN meet AB at E. From $\triangle$s ANC, ANE, CN = NE and AC = AE $\therefore \dfrac{CN}{NE} = \dfrac{CO}{OB}$ $\therefore$ ON is $\parallel$ to BE (V. 2.). Also $\dfrac{ON}{BE} = \dfrac{CO}{CB} = \dfrac{1}{2}$ $\therefore ON = \frac{1}{2}BE = \frac{1}{2}(AB - AC)$ for AC = AE.

25. Let P be the centre of circle ABC, Q that of circle ABD. Let Q fall on the opp. side of BD to the pt. A, so that P falls

on the same side of BC as the pt. A. $\angle$BQD = 2 supplement of $\angle$BAD (III. 12.) = 2$\angle$CAB = $\angle$CPB $\therefore$ $\angle$PCB + $\angle$PBC = $\angle$QDB + $\angle$QBD (I. 22.), *i.e.* $\angle$PCB = $\angle$PBC = $\angle$QDB = $\angle$QBD $\therefore$ $\triangle$s PCB, QDB are equiangular $\therefore$ $\frac{BC}{BD} = \frac{PC}{QB} = \frac{\text{diameter of circle ABC}}{\text{diameter of circle ABD}}$.

26. $\angle$EAC = $\frac{1}{2}(180° - A)$ and $\angle$ECA = $\frac{1}{2}(180° - C)$ $\therefore$ $\angle$AEC = $180° - \frac{1}{2}(180° - A) - \frac{1}{2}(180° - C) = \frac{A + C}{2} = \frac{180° - B}{2}$ = $\angle$FBA. Also $\angle$F is common to $\triangle$s FAB, FDE $\therefore$ the $\triangle$s are equiangular (I. 22.) $\therefore$ $\frac{FA}{FB} = \frac{FD}{FE}$, *i.e.* FA and FB are inversely proportional to FE and FD.

27. Let N be the mid. pt. of BC. NE bisects the arc BC of the circum-circle of $\triangle$ ABC. Also since $\angle$BAE = $\angle$CAE, ADE also bisects this arc $\therefore$ E lies on the circum-circle at the mid. pt of arc BC. In $\triangle$s ABE, BDE, $\angle$BEA is common. $\angle$BAE = $\angle$EAC = $\angle$DBE in the same segment $\therefore$ the $\triangle$s are equiangular $\therefore$ $\frac{AE}{EB} = \frac{EB}{ED}$ $\therefore$ rect. AE . ED = BE^2.

28. $\triangle$s FED, AEB are equiangular (I. 20.) $\therefore$ $\frac{FD}{AB} = \frac{DE}{EB} = \frac{3}{1}$ $\therefore$ FD = 3AB, *i.e.* FC + CD = 3AB $\therefore$ FC = 2AB (II. 2.).

EXERCISES LVII.

1. $\frac{AO}{OD} = \frac{CO}{OB}$ $\therefore$ rect. AO . OB = rect. CO . OD $\therefore$ A, B, C, D are concyclic (IV. 13.).

2. $\frac{OA}{OC} = \frac{OD}{OB}$ $\therefore$ rect. OA . OB = rect. OC . OD. If the circum-circle of $\triangle$BAC does not pass thro. D, let it cut OC again at E.

Then rect. OE . OC = rect. OA . OB (IV. 14. Cor.) = rect. OC . OD $\therefore$ OD = OE, *i.e.* D must coincide with E.

3. $\frac{BD}{DA} = \frac{DA}{DC}$ and $\angle$BDA = $\angle$CDA $\therefore$ $\triangle$BDA is similar to $\triangle$ADC (V. 7.) $\therefore$ $\angle$BAC = $\angle$BAD + $\angle$DAC = $\angle$DCA + $\angle$ABD $\therefore$ $\angle$BAC = a rt. $\angle$ (I. 22.).

4. $\triangle$s ADE, ABC are similar by V. 7. $\therefore$ $\angle$DAC $=$ $\angle$DAE $\therefore$ AE falls along AC, *i.e.* AEC is a str. line.

5. Draw BE perp. to AC. $\frac{AC}{CD}=\frac{AB}{BC}$ $\therefore$ rect. AC . BC $=$ rect. AB . CD. But $\frac{1}{2}$AB . CD $=$ area of $\triangle$ABC $=\frac{1}{2}$AC . BE $\therefore$ $\frac{1}{2}$AC . BE $=\frac{1}{2}$AC . BC $\therefore$ BE $=$ BC. But BE is perp. to AC $\therefore$ BE must coincide with BC, *i.e.* $\angle$ACB $=$ a rt. $\angle$ $\therefore$ $\angle$BCD $=$ complement of $\angle$ACD $=$ $\angle$CAB.

6. BC $=$ BD $\therefore$ $\angle$BCD $=$ $\angle$BDC $\therefore$ $\angle$ACB $=$ $\angle$BDE. Hence in $\triangle$s ACB, BDE, $\frac{AC}{CB}=\frac{BD}{DE}$ and $\angle$ACB $=$ $\angle$BDE $\therefore$ $\triangle$s ACB, BDE are similar (V. 7.) $\therefore$ $\angle$CBA $=$ $\angle$DEB, and $\angle$A is common to $\triangle$s ACB, ABE $\therefore$ $\triangle$s ACB, ABE are equiangular (I. 22.) and $\therefore$ similar (V. 5.).

7. $\frac{AE}{AC}=\frac{DF}{DA}$ (hyp.) $\therefore$ $\frac{AE}{AB}=\frac{DF}{DB}$. Also $\angle$EAB $=$ $\angle$BDF $\therefore$ $\triangle$s AEB, DFB are similar (V. 7.) $\therefore$ $\angle$EBA $=$ $\angle$FBD $\therefore$ adding $\angle$ABF to each, $\angle$EBF $=$ $\angle$ABD. Also since $\triangle$s AEB, DFB are similar, $\frac{AB}{EB}=\frac{BD}{BF}$, *i.e.* $\frac{AB}{BD}=\frac{EB}{BF}$ and $\angle$ABD $=$ $\angle$EBF $\therefore$ $\triangle$s EBF, ABD are similar.

8. Let the lines thro. A and B meet in Q. Join CQ. $\triangle$s AQB, DPE have their sides respectively parallel and are $\therefore$ similar $\therefore$ $\frac{BQ}{EP}=\frac{AB}{DE}=\frac{2}{1}=\frac{BC}{EF}$ $\therefore$ $\frac{BQ}{BC}=\frac{EP}{EF}$. Also QB and BC are respectively $\parallel$ to EP and EF $\therefore$ $\angle$QBC $=$ $\angle$PEF, and as proved above, $\frac{BQ}{BC}=\frac{EP}{EF}$ $\therefore$ $\triangle$QBC is similar to $\triangle$PEF (V. 7.) $\therefore$ QC is $\parallel$ to PF, which proves the proposition.

9. Take O the centre of the circle on which the pt. P lies, and let C be one of the cutting pts. of the circles. Rect. OA . OB $=$ OC^2 (IV. 14.) (for OC is a tangent to the circle) ABC $=$ OP^2 $\therefore$ $\frac{OA}{OP}=\frac{OP}{OB}$ and $\angle$O is common to $\triangle$s PAO, BPO $\therefore$ these $\triangle$s are similar (V. 7.) $\therefore$ $\frac{PA}{PB}=\frac{PO}{OB}=$ a constant ratio.

10. In $\triangle$s DBC, PAC, $\angle DBC = \angle PAC$ and $\frac{DB}{BC} = \frac{PA}{AC}$ $\therefore$ the $\triangle$s are similar $\therefore$ $\angle DCB = \angle PCA$ $\therefore$ adding $\angle BCP$ to each, $\angle DCP = \angle BCA = \frac{1}{2}$ a rt. $\angle$. Also since $\triangle$s DCB, PCA are similar, $\frac{DC}{PC} = \frac{BC}{CA}$ and the included $\angle$s DCB, PCA are equal $\therefore$ $\triangle DPC$ is similar to $\triangle BAC$ $\therefore$ $\angle DPC = \angle BAC =$ a rt. $\angle$ $\therefore$ $\angle PDC = \frac{1}{2}$ a rt. $\angle$ (I. 22.) $\therefore$ $PD = PC$.

11. Let DE produced meet AC at F. Draw BH $\parallel$ to AC to meet DE at H. From similar $\triangle$s DBH, DAF, $\frac{BH}{AF} = \frac{DB}{DA}$ (V. 5.) $= \frac{BE}{EC}$ (hyp.) $= \frac{BH}{CF}$ from similar $\triangle$s HEB, FEC $\therefore$ $AF = CF$.

12. In $\triangle$s ABC, DBA, $\angle B$ is common, and $\frac{BD}{BA} = \frac{BA}{BC}$ $\therefore$ the $\triangle$s are similar (V. 7.) $\therefore$ $\frac{BD}{AD} = \frac{BA}{AC}$, and $\angle BDA = \angle CAB$. In like manner $\triangle$s AEC, BAC are similar, and $\therefore$ $\frac{AE}{EC} = \frac{BA}{AC}$ and $\angle AEC = \angle BAC$ $\therefore$ $\angle AEC = \angle BDA$ $\therefore$ $\angle ADE = \angle AED$ $\therefore$ $AE = AD$ $\therefore$ $\frac{BD}{AD} = \frac{BA}{AC} = \frac{AE}{EC} = \frac{AD}{EC}$.

13. Rect. $OA . OB = OC^2$ (IV. 14.) $= OD^2$ $\therefore$ $\frac{OA}{OD} = \frac{OD}{OB}$. Also $\angle BOD$ is common to $\triangle$s DOA, BOD $\therefore$ these $\triangle$s are similar (V. 7.) $\therefore$ $\angle ODA = \angle OBD = \angle CEA$ (in same segment) $\therefore$ EF is $\parallel$ to OD (I. 20.).

14. Rect. $AO\ BO = OC^2$ (hyp.) $= OP^2$ $\therefore$ $\frac{AO}{OP} = \frac{OP}{OB}$ and $\angle O$ is common to $\triangle$s AOP, POB $\therefore$ these $\triangle$s are similar (V. 7.) $\therefore$ $\angle BPO = \angle OAP$. Also $\angle CPA + \angle PAC = \angle OCP$ (I. 22.) $= \angle OPC = \angle BPC + \angle BPO$ $\therefore$ $\angle CPA = \angle BPC$.

15. Let the line thro. C meet AD at G and AE at H. From similar $\triangle$s CDG, BDA, $\frac{CG}{BA} = \frac{CD}{DB} = \frac{CA}{AB}$ (V. 3.) $= \frac{EC}{EB}$ (V. 4.) $= \frac{CH}{AB}$ from similar $\triangle$s EBA, ECH $\therefore$ $CG = CH$.

EXERCISES LVIII.

1. Let ABC be an isos. $\triangle$ having AB = AC, and let AD be perp. to the base BC. Produce AD to meet the circum-circle in E. Join BE. The centre of the circum-circle lies in AD $\therefore$ ABE is a rt. $\angle$. Also AD is perp. to BC $\therefore$ $\triangle$s ABE, ADB are similar (V. 9.) $\therefore \dfrac{EA}{AB} = \dfrac{AB}{AD}$.

2. Let ABC be the $\triangle$ rt. $\angle$d. at C, so that AB = 10, and AC = 7. Draw CD perp. to AB. $\triangle$s ACB, ADC are similar (V. 9.) $\therefore \dfrac{AD}{AC} = \dfrac{AC}{AB}$ $\therefore$ $AD = \dfrac{49}{10} = 4{\cdot}9$, and $BD = 10 - 4{\cdot}9 = 5{\cdot}1$.

EXERCISES LIX.

1. ACD is a str. line (I. 2.). $\triangle ABC = \triangle DCE$ $\therefore$ rect. AC . CB = rect. DC . CE $\therefore \dfrac{AC}{CD} = \dfrac{CE}{CB}$ $\therefore \dfrac{AC}{AC + CD} = \dfrac{CE}{CE + CB}$, *i e.* $\dfrac{AC}{AD} = \dfrac{EC}{EB}$. From similar $\triangle$s AFC, ABD, $\dfrac{CF}{DB} = \dfrac{CA}{DA}$ (V. 5.) $= \dfrac{EC}{EB} = \dfrac{CG}{DB}$ from similar $\triangle$s ECG, EBD $\therefore$ CF = CG.

2. Join AD, BE. $\angle ADC = \angle ACD = \angle ECB = \angle BEC$ $\therefore$ $\triangle$s DAC, EBC are equiangular $\therefore \dfrac{AC}{CD} = \dfrac{BC}{CE}$ (V. 5.) $\therefore$ rect. AC . CE = rect. BC . CD $\therefore$ $\triangle ACE = \triangle DCB$ (V. 10.).

3. Rect. OD . OC = rect. OP . OQ (IV. 14. Cor.) = rect. OA . OB (IV. 14. Cor.) $\therefore$ rect. OD . OC is constant for all directions of OPQ, and C is a fixed pt. $\therefore$ D is a fixed pt. Also since rect. OA . OB = rect. OC . OD, $\dfrac{OB}{OC} = \dfrac{OD}{OA}$, *i.e.* $\dfrac{OC + CB}{OC} = \dfrac{OA + AD}{OA}$ $\therefore \dfrac{CB}{OC} = \dfrac{AD}{OA}$, *i.e.* $\dfrac{OA}{AD} = \dfrac{OC}{CP}$.

4. From similar $\triangle$s GBA, ABC, $\dfrac{BG}{BA} = \dfrac{BA}{BC}$ (V. 9.) $\therefore$ rect. $BG . BC = BA^2$, *i.e.* rect. BG . BF = rect. BE . BA. Also $\angle FBG = \frac{2}{3}$ of a rt. $\angle = \angle ABE$ $\therefore$ $\triangle BFG = \triangle BEA$. In like manner, $\triangle FGC = \triangle CDA$.

5. Draw CD perp. to the hypotenuse AB of the $\triangle$, and let $BC = 2CA$. The circles on diameters BC, CA each pass thro. D $\therefore$ CD is their common chord. Also rect. $CD . AB = 2$ area of $\triangle ABC =$ rect. $CA . CB = 2CA^2$ $\therefore$ $\frac{CD}{AB} = \frac{2CA^2}{AB^2} = \frac{2CA^2}{BC^2 + CA^2} = \frac{2}{5}$, for $BC^2 = 4CA^2$ $\therefore$ $CD = \frac{2}{5}AB$.

6. In $\triangle$s ABC, CBD $\angle B$ is common, $\angle ACB = \angle BDC$ $\therefore$ the $\triangle$s are equiangular $\therefore$ $\frac{BA}{BC} = \frac{BC}{BD}$ $\therefore$ rect. $BA . BD = BC^2$ $\therefore$ BC touches the circum-circle of $\triangle ADC$ (IV. 15.).

7. If AB is the common chord, $OP . OQ = OA . OB$ (IV. 13.) $= OR . OS$ (IV. 13) $\therefore$ $\frac{OP}{OR} = \frac{OS}{OQ}$.

8. $\angle AEC =$ a rt. $\angle = \angle DOC$ $\therefore$ DOEC are concyclic $\therefore$ rect. $AD . AE =$ rect. $AO . AC = 2AO^2 = AB^2$ $\therefore$ AB touches the circum-circle of $\triangle BDE$ (IV. 15.).

9. Draw AK to touch the circle at K. Rect. $AC . AB = AK^2$ (hyp.) $=$ rect. $AE . AD$ (IV. 14.) $\therefore$ C, B, D, E are concyclic $\therefore$ $\angle CBE = \angle CDE$ (in the same segment) $= \angle DFE$ (III. 18.) $\therefore$ DF is $\parallel$ to AB (I. 18.).

10. In $\triangle$s ABD, BOD, $\angle OBD = \angle CAD$ (in same segment) $= \angle BAD$, and $\angle BDO$ is common $\therefore$ the $\triangle$s are equiangular $\therefore$ $\frac{AD}{DB} = \frac{DB}{DO}$ $\therefore$ $DB^2 =$ rect. $AD . DO$.

11. $\frac{\triangle AOC}{\triangle BOD} = \frac{\text{rect. } AO . OC}{\text{rect. } BO . OD}$ (V. 10) $= \frac{3}{4} \times \frac{2}{5} = \frac{3}{10}$. $\frac{\triangle AOD}{\triangle BOC} = \frac{\text{rect. } AO . OD}{\text{rect. } OB . OC} = \frac{3}{4} \times \frac{5}{2} = \frac{15}{8}$.

12. $\triangle AOC = \triangle BOD$ $\therefore$ rect. $AO . OC =$ rect. $DO . OB$ $\therefore$ $4 \times 5 = 2OD$ $\therefore$ $OD = 10$.

EXERCISES LX.

1. Let A be the area reqd. $\frac{A}{75} = \frac{3^2}{5^2}$ (V. 11.) $\therefore$ $A = 27$ sq. in.

2. $\triangle$s ADE, ABC are similar $\therefore$ $\frac{\triangle ADE}{\triangle ABC} = \frac{AD^2}{AB^2}$ (V. 11.) $= \frac{3^2}{8^2} = \frac{9}{64}$.

3. Let DE be the reqd. line, meeting AB produced at D and AC produced at E. $\frac{AB^2}{AD^2}=\frac{\triangle ABC}{\triangle ADE}=\frac{1}{4}$ $\therefore$ $AD=2AB$.

4. If DE is the reqd. line, meeting AB and AC produced at D and E, $\frac{AB^2}{AD^2}=\frac{\triangle ABC}{\triangle ADE}$ (V. 11.) $=\frac{1}{9}$ $\therefore$ $AD=3AB$.

5. $\frac{\triangle ADE}{\triangle ABC}=\frac{AD^2}{AB^2}$ (V. 11.) $=\frac{1}{4^2}=\frac{1}{16}$.

6. $\frac{AD^2}{AB^2}=\frac{\triangle ADE}{\triangle ABC}$ (V. 11.) $=\frac{1}{9}$ $\therefore$ $\frac{AD}{AB}=\frac{1}{3}$, *i.e.* $\frac{AD}{AD+DB}=\frac{1}{3}$ $\therefore$ $\frac{AD}{BD}=\frac{1}{2}$.

7. $\angle BDA$ = a rt. $\angle = \angle BEA$ $\therefore$ B, D, E, A are concyclic $\therefore$ $\angle DEC$ = supplement of $\angle AED = \angle ABD$ (III. 13.). Also $\angle C$ is common to $\triangle$s DEC, ABC $\therefore$ the $\triangle$s are equiangular (I. 22.) $\therefore$ $\frac{\triangle CDE}{\triangle ABC}=\frac{CD^2}{CA^2}$ (V. 11.).

8. $\frac{\triangle BDF}{\triangle FAD}=\frac{BF}{FA}$ (V. 1.) $=\frac{BH}{HD}$ (V. 2.). But $\triangle AFD=\frac{1}{2}HD.AD$ $\therefore$ $\triangle BDF=\frac{1}{2}BH.AD$. Similarly $\triangle AED=\frac{1}{2}DG.AD$ and $\triangle DEC=\frac{1}{2}CG.AD$ $\therefore$ quadl. $AFDE=\frac{1}{2}HG.AD$ $\therefore$ the three figures are proportional to BH, HG, GC.

9. From similar $\triangle$s BFD, BAC, $\frac{BF}{FD}=\frac{BA}{AC}$ $\therefore$ $\frac{BF}{AE}=\frac{BA}{AC}$ ... (1). Also $\frac{AE}{EC}=\frac{BD}{DC}$ (V. 2.) $=\frac{BA}{AC}$ (V. 3.) $\therefore$ from (1) $\frac{BF}{EC}=\frac{BA^2}{AC^2}$.

10. If ABCD is a sq., $\frac{\triangle \text{ on } AB}{\text{similar } \triangle \text{ on } BD}=\frac{AB^2}{BD^2}$ (V. 11.) $=\frac{AB^2}{2AB^2}$ (II. 11.) $\therefore$ $\triangle$ on $AB=\frac{1}{2}$ similar $\triangle$ on BD.

11. Let DEF be the equilateral $\triangle$ formed by the perpendiculars, A lying in EF, B in DF, C in DE. $\triangle DBC$ is half the equilateral $\triangle$ on CD $\therefore$ $BD=\frac{1}{2}CD$. Also $CE=DB$ $\therefore$ $DE=3DB$. Also from $\triangle DBC$, $DC^2=DB^2+BC^2$ (II. 11.) $\therefore$ $4DB^2=DB^2+BC^2$ $\therefore$ $3DB^2=BC^2$. $\frac{\triangle ABC}{\triangle DEF}=\frac{BC^2}{DE^2}=\frac{BC^2}{9DB^2}=\frac{3DB^2}{9DB^2}=\frac{1}{3}$.

12. If the $\triangle BDC$ were folded about BC, the pt. D would coincide with G the centroid of $\triangle BAC$ $\therefore$ $\triangle BDC = \frac{1}{3}\triangle BAC$.

13. Let DE drawn $\parallel$ to AB meet AC at E. $\frac{\triangle DEC}{\triangle ABC} = \frac{CD^2}{CB^2}$ (V. 11.) $= \frac{CD^2}{2CD^2} = \frac{1}{2}$.

14. $\angle EBD$ = supplement of $\angle ABD = \angle ACD$ (III. 13.). Also $\angle E$ is common to $\triangle$s EBD, ECA $\therefore$ $\triangle$s EBD, ECA are equiangular $\therefore \frac{\triangle EAC}{\triangle EBD} = \frac{AC^2}{BD^2}$ (V. 11.).

15. Let BAC be a $\triangle$ rt. $\angle$d. at A, and let AD be drawn perp. to BC. $\frac{BD}{DC} = \frac{\triangle ABD}{\triangle CAD}$ (V. 1.) $= \frac{AB^2}{AC^2}$ (V. 11.), for $\triangle$s ABD, CAD are similar (V. 9.).

16. In $\triangle$s ANC, BAC, $\angle C$ is common, $\angle ANC = \angle BAC$ $\therefore$ the $\angle$s are equiangular. In like manner, $\triangle$s AMB, CAB are equiangular $\therefore$ $\triangle$s AMB, ANC are equiangular. Also $\frac{BM}{NC} = \frac{\triangle AMB}{\triangle ANC}$ (V. 1.) $= \frac{AB^2}{AC^2}$ (V. 11.).

17. OL and OM are respectively perp. to AB and BC. Join OB cutting AM at P. From $\triangle$s LOB, MOB, $\angle LOB = \angle MOB$ (I. 7.) $\therefore$ from $\triangle$s LOP, MOP, $\angle LPO = \angle MPO$ = a rt. $\angle$ $\therefore$ $\frac{\triangle LBM}{\triangle OLM} = \frac{BP}{OP}$ (V. 1. Cor.) $= \frac{\triangle LPB}{\triangle LPO}$ (V. 1.) $= \frac{BL^2}{OL^2}$ (V. 11.), (for $\triangle$s LPB, LPO are similar) $= \frac{(\text{side of fig.})^2}{(\text{diamr. of in-circle})^2}$.

18. The rt. $\angle$d. $\triangle$s OPN, OTC have the $\angle O$ common $\therefore$ the $\triangle$s are similar $\therefore$ $\frac{\triangle OPN}{\triangle OTC} = \frac{OP^2}{OT^2}$ (V. 11.) $= \frac{OP^2}{OP \cdot OQ}$ (IV. 14.) $= \frac{OP}{OQ}$.

19. Join AC. $\frac{\triangle BFG}{\triangle BAC} = \frac{BF^2}{BA^2}$ (V. 11.) $= \frac{1}{9}$ $\therefore$ $\triangle BFG = \frac{1}{9}\triangle BAC$. In like manner, $\triangle DML = \frac{1}{9}\triangle DAC$ $\therefore$ $\triangle BFG + \triangle DML = \frac{1}{9}$ fig. ABCD. In like manner, $\triangle AEN + \triangle CHK = \frac{1}{9}$ fig. ABCD $\therefore$ fig. EFGHKLMN $= \frac{7}{9}$ of fig. ABCD.

20. Let O be the centre of the circle, AB a side of the inscribed hexagon; EAF, EBG sides of the circum-hexagon. Join EO meeting AB at H. As in Question 12, $\triangle AEB = \frac{1}{3}\triangle AOB$ $\therefore$ quadl. $OAEB = \frac{4}{3}\triangle AOB$ $\therefore$ the circumscribed hexagon $= \frac{4}{3}$ of the inscribed.

21. $EA = EC$ $\therefore$ $\angle EAC = \angle ECA = \angle ABC$ (III. 18.) $= \angle OCB$ $\therefore$ $\triangle$s EAC, OCB are equiangular $\therefore$ $\frac{\triangle ECA}{\triangle OCB} = \frac{AC^2}{BC^2}$ (V. 11.) $= \frac{\triangle ADC}{\triangle CDB}$ (V. 11.) for these $\triangle$s are similar (V. 9.) $= \frac{AD}{BD}$ (V. 1.).

EXERCISES LXI.

1. Let EG meet BC at K. $\triangle$s CKG, HAE, EKB are equiangular (I. 2. and 20.). Also CG, AE, EB are corresponding sides in these $\triangle$s. And $AE = CF$ and $EB = FG$ $\therefore$ by V. 15. $\triangle EBK = \triangle AEH + \triangle CKG$. Adding DAEKC to each, parm. ABCD $= \triangle DHG$.

EXERCISES LXII.

1. Let the circles ACB, ADB intersect in A and B, P and Q being their centres Let PQ meet the circles in C and D between P and Q. Draw the diameter EQF perp. to PQ. $\angle QAP = \angle QBP =$ a rt. $\angle$ $\therefore$ $\angle$s AQP, APQ, EQA, BQP are each equal to half a rt. $\angle$ $\therefore$ half the circle $ADB = 4$ sector AQD $= 2$ sector AQB = sector AQB + sector APB = fig. AQBP + area common to the circles $= PA^2 +$ area common to both circles.

EXERCISES LXIII.

1. $AD^2 = BD \cdot DC = AD \cdot DE$ (IV. 13.) $\therefore$ $DA = DE$. Also in $\triangle$s EDC, ECA, $\angle E$ is common, $\angle ECD = \angle BAE$ (in same segment) $= \angle EAC$ $\therefore$ the $\triangle$s are equiangular $\therefore$ $\frac{ED}{EC} = \frac{EC}{EA}$ $\therefore$ $EC^2 = ED \cdot EA$ $\therefore$ $2EC^2 = 2 \cdot ED \cdot EA = EA^2$.

2. AD bisects $\angle A$ (V. 3.). As in the previous exercise $BE^2 = AE \cdot ED$. Also as in V. 17., $AB \cdot AC = AE \cdot AD$ $\therefore$ $AB \cdot AC + BE^2 = AE \cdot AD + AE \cdot ED = AE^2$.

3. If P is the reqd. pt., and PM, PN be drawn perp. to AC and BD, and if X is the diameter of the circle, $PA \cdot PC = X \cdot PM$

and PB . PD = X . PN $\therefore$ PM = PN. Hence if AC and BD meet at O we bisect either of the $\angle$s AOB, AOD, any one of the four points where these lines meet the circumference will satisfy the reqd. condition.

4. By V. 19. PB . AC + PC AB = PA . BC $\therefore$ PB + PC = PA.

5. Let the bisector of $\angle$A in $\triangle$ABC meet BC at D and the circum-circle at E. As in Example 1 above $EC^2 = ED . EA = 2ED^2$ for EA = 2 . ED. Also from the similar $\triangle$s EDC, ECA, $\frac{CA}{CE} = \frac{DC}{DE}$ $\therefore$ $\frac{CA^2}{CD^2} = \frac{CE^2}{DE^2} = 2$, *i.e* $CA^2 = 2CD^2$. In like manner, $BA^2 = 2BD^2$.

6. Let AB be the given base, P any position of the vertex, and let PM be perp. to AB. The rect. PA . PB $\propto \frac{1}{2}$PM . AB. Let X be the diameter of the circum-circle of $\triangle$PAB. Then by V. 18. PA . PB = X . PM $\therefore$ X . PM $\propto \frac{1}{2}$PM . AB $\therefore$ X is constant $\therefore$ the locus of P is a circle which passes thro. the pts. A and B.

7. Let ABCD be the quadl., and let its diagonals meet at O. Draw BM, DN perp. to AC. BA . BC = X . BM where X is the diameter of the circum-circle (V. 18). Also DA . DC = X . DN by the same prop. $\therefore$ $\frac{BA . BC}{DA . DC} = \frac{BM}{DN} = \frac{OB}{OD}$ from the similar $\triangle$s BOM, DON.

8. Let EB produced meet the circum-circle of $\triangle$ABC at F. Join AF. $\angle$AFB = supplement of $\angle$ACB = $\angle$BCE. $\angle$ABF = $\angle$DBE = $\angle$CBE $\therefore$ $\triangle$s ABF, EBC are equiangular $\therefore$ $\frac{AB}{BF} = \frac{BE}{BC}$ (V. 5.) $\therefore$ AB . BC = BE . BF $\therefore$ AB . BC + BE^2 = BE^2 + BE . BF = BE . EF (IV. 3.) = CE . EA (IV. 14. Cor.).

9. Let ABCD be the quadl., AC being a diameter of the circum-circle, and AC bisecting BD at E. AC is at rt. $\angle$s to BD (III. 3.). AB . CD + AD . BC = AC . BD (V. 19.) = AC . BE + AC . DE = 2$\triangle$ABC + 2$\triangle$ADC = 2 fig. ABCD.

10. Draw AM perp. to BC. Let X be the diameter of the circum-circle of $\triangle$ABD, and Y that of the circum-circle of $\triangle$ACD. AB . AD = X . AM (V. 18.). AC . AD = Y . AM (V. 18.) $\therefore$ $\frac{X}{Y} = \frac{AB}{AC} = \frac{AD}{CD}$ from the similar $\triangle$s ABD, CAD.

EXERCISES LXIV.

1. Let AB be the given str. line. Draw any other str. line AC, and from it cut off equal parts AD, DE, EF, FG, GH. Join BH, and draw DP, EQ, FR, GS || to BH to meet AB at P, Q, R, S. By V. 21. AB is divided into 5 equal parts at P, Q, R, S.

2. Let AB be the given line. Draw any other line AH from it, and cut off equal parts AD, DE, EF, FG, GH. Join EB and draw HP || to EB to meet AB produced in P. $\frac{AP}{PB}=\frac{AH}{HE}$ (V. 2.) $=\frac{5}{3}$.

3. Let EF be the given line, ABCD the given rect. To EF AB, BC, find a fourth proportional FG as in V. 23. $\frac{EF}{AB}=\frac{BC}{FG}$ $\therefore$ $EF \cdot FG = AB \cdot BC$ $\therefore$ the rect. contained by EF and FG is the reqd. rect.

4. If ABCD is the given rect., to AB and BC find a mean proportional EF as in V. 24. $EF^2 = AB \cdot BC$ $\therefore$ the sq. on EF is the reqd. sq.

5. $\angle ABC$ = supplement of $\angle CBE = \angle CDE$, and $\angle A$ is common to $\triangle$s ABC, ADE $\therefore \frac{AE}{AD}=\frac{AC}{AB}=\frac{4}{3}$. Also $\frac{AD}{AB}=\frac{10}{3}$ $\therefore \frac{AE}{AB}=\frac{40}{9}$, *i.e.* $\frac{AB+BE}{AB}=\frac{40}{9}$ $\therefore \frac{BE}{BA}=\frac{31}{9}$.

6. Draw AB 3·6 in. long, and in AB produced make BC 1 in. long. On AC describe a semi-circle ADC, and draw BD perp. to ABC to meet the circle at D. $BD^2 = AB \cdot BC$ (V. 9. Cor.) $= 3{\cdot}6$ sq. in. $\therefore$ the sq. on BD is the reqd. sq. By measurement $BD = 1{\cdot}90$ in.

7. Draw AB 2·4 in. long, and draw AC perp. to AB and 1 in. long. Join BC. Draw CD perp. to BC to meet BA pro-

duced in D. $\triangle$s CAB, DAC are similar (V. 9.) $\therefore \frac{AB}{AC}=\frac{AC}{AD}$ $\therefore$ $AB \,.\, AD = AC^2 = 1$ sq. in. $\therefore$ the rect. contained by AB and AD is the reqd. rect.

8. Draw lines 3·7, 1·7, and 2·9 in. long. Find AB a fourth proportional to these. Then $\frac{3 \cdot 7}{1 \cdot 7}=\frac{2 \cdot 9}{AB}$ $\therefore$ $AB \times 3 \cdot 7 = 1 \cdot 7 \times 2 \cdot 9$ $\therefore$ the rect. whose sides are AB and the line 3·7 in. long is the reqd. rect.

9. See Book V., Prop. 28.

10. If ABC is the given $\triangle$, bisect BC at D, and draw BF, DE perp. to BC. Also draw AEF || to BC. Rect. FBDE = $\triangle$ABC (IV. 9.). To BD and DE find a mean proportional X (V. 24.). Then $BD \,.\, DE = X^2$ $\therefore$ the sq. on X is the reqd. sq.

11. Use the method of Example 6 above. By measurement the side of the sq. will be found to be 1·67 in.

12. Describe, by the method of Example 6 above, a sq. whose area is 3·6 sq. in. If X be its side, $X^2 = 3 \cdot 6$ $\therefore$ $X = \sqrt{3 \cdot 6}$. By measurement X will be found to be 1·90 in. long $\therefore$ $\sqrt{3 \cdot 6} = 1 \cdot 90$.

13. Describe the sq. by the method of Example 10 above. By measurement its side will be found to be 1·49 in.

14. Let ABC be the given $\triangle$. Trisect AB at D and E. (V. 20.). Draw DH || to BC to meet AC at H. $\triangle$s ADH, ABC are equiangular (I. 20. and 22.) $\therefore$ $\frac{\triangle ADH}{\triangle ABC}=\frac{AD^2}{AB^2}=\frac{1}{9}$ $\therefore$ ADH is the reqd. $\triangle$.

15. Let ABC be the given $\triangle$. Take $AD = \frac{1}{4}AB$ (V. 20.). Draw DE || to BC to meet AC at E. As in the preceding example $\frac{\triangle ADE}{\triangle ABC}=\frac{AD^2}{AB^2}=\frac{1}{16}$ $\therefore$ ADE is the reqd. $\triangle$.

16. Produce AB to D, making BD equal to BA. Draw DE || to AC and CE || to AD. Parm. $ADEC = 2\triangle ADC = 4\triangle ABC$ $\therefore$ ADEC is the reqd. parm.

17. From the external pt. A draw any str. line cutting the circle at B and C. Take X a mean proportional to AB, AC. With centre A and rad. X describe a circle cutting the given circle at D. $AC . AB = X^2 = AD^2$ $\therefore$ AD is a tangent to the given circle (IV. 15.).

18. Draw AB 3·12 in. long, and BC at rt. $\angle$s to it 1·28 in. long, and complete the rect. ABCD. To AB and BC find a mean proportional BE as in V. 24. $BE^2 = AB . BC$ and the sq. on BE is the sq. reqd. By measurement BE = 2 in., and the diagonal of the sq. on BE = 2·82 in.

19. Draw $\angle BAC = 30°$ with a protractor, or by bisecting the $\angle$ of an equilateral $\triangle$. Make AB = 4 in. and AC = 5 in. Join CB. Produce AB to D making AD = 4·5 in. Join CD, and draw BE $\parallel$ to DC, to meet AC at E. Join DE. $\triangle BEC = \triangle BED$ (II. 5.) $\therefore$ $\triangle ADE = \triangle ABC$. Thro. E draw EF $\parallel$ to AD. With centre A and rad. 3·25 in. describe a circle cutting EF at F. Join AF. DF. $\triangle ADF = \triangle AED$ (II. 5.) $= \triangle ABC$ $\therefore$ ADF is the $\triangle$ reqd.

20. Use the method of Example 10 above.

21. By V. 22. find E a third proportional to A and B Then $\frac{A}{E} = \frac{A^2}{B^2}$ (V. 12., Cor. 2.). To A, E, and C find a fourth proportional D, as in V. 23. Then $\frac{C}{D} = \frac{A}{E} = \frac{A^2}{B^2}$ $\therefore$ D is the reqd. line.

22. Let AB be the given line. With centre B and any rad describe a circle; and with centre A and rad. double the first describe a second circle, cutting the first at C. Join AC, BC, and bisect $\angle ACB$ by AD meeting AB at D. $\frac{AD}{DB} = \frac{AC}{CB}$ (V. 3.) $= \frac{2}{1}$ $\therefore$ D is a point of trisection of AB. Bisecting AD we have the other pt. of trisection.

23. Let AB be the given str. line. Draw any other str. line AC, and in it make AD = 5 cms. and DE = 3 cms. Join EB, and draw DF $\parallel$ to EB to cut AB at F. $\frac{AF}{FB} = \frac{AD}{DE}$ (V. 2.) $= \frac{5}{3}$.

24. Let AB be the given line. With any unit greater than one-fifth of AB, describe a circle with centre A and rad. 3 units. Also with centre B and rad. 2 units describe a circle cutting the first at C. Join AC, BC, and produce AC to D. Bisect $\angle$BCD by CE meeting AB produced at E. $\frac{AE}{EB}=\frac{AC}{CB}$ (V. 4.) $=\frac{3}{2}$.

25. Let ABC be the $\triangle$. Bisect AB at D, and draw DE perp. to AB, making DE = DA = DB. With centre A and rad. AE describe a circle cutting AB at F. Draw FG $\parallel$ to BC, meeting AC at G. From $\triangle$s BDE, ADE, $\angle BED = \angle EBD = \angle EAD = \angle DEA = \frac{1}{2}$ a rt. $\angle$, and $BE = AE$ $\therefore$ $BA^2 = 2EA^2$ (II. 11.) $= 2AF^2$ $\therefore$ $\frac{\triangle AFG}{\triangle ABC} = \frac{AF^2}{AB^2} = \frac{1}{2}$ $\therefore$ $\triangle AFG = \frac{1}{2}\triangle ABC$.

26. Let ABC be the $\triangle$. On AB describe the equilateral $\triangle$ABD. Bisect $\angle$ABD by BE, and draw AE perp. to BA to meet this line at E. With centre A and rad. AE describe a circle cutting AB at F. Draw FG $\parallel$ to BC to meet AC at G. $\angle ABE = 30°$ and $\angle BAE$ = a rt. $\angle$ $\therefore$ $\triangle$BEA is half the equilateral $\triangle$ on BE $\therefore$ $BE = 2EA$ $\therefore$ $AB^2 = BE^2 - EA^2 = 3EA^2$. $\frac{\triangle AFG}{\triangle ABC} = \frac{AF^2}{AB^2} = \frac{AE^2}{AB^2} = \frac{1}{3}$ $\therefore$ $\triangle AFG = \frac{1}{3}\triangle ABC$. Again, at F draw FH perp. to AB and equal to FA. With centre A and rad. AH describe a circle cutting AB at K. Draw KM $\parallel$ to BC to meet AC at M. $\frac{\triangle AKM}{\triangle AFG} = \frac{AK^2}{AF^2} = \frac{AH^2}{AF^2} = 2$ $\therefore$ $\triangle AKM = 2\triangle AFG = \frac{2}{3}\triangle ABC$ $\therefore$ FG and KM trisect triangle ABC.

27. Draw AB so that the sq. on AB = the given sum of sqs. On AB as diamr. describe a circle, centre O, and draw OD perp. to AB to meet the circle at D. Divide AB in the given ratio at C. Join DC, and produce it to meet the circle at E. Join AE, BE. $\angle AEC = \frac{1}{2}\angle AOD = \frac{1}{2}$ a rt. $\angle$. In like manner $\angle DEB = \frac{1}{2}$ a rt. $\angle$ $\therefore$ $AE^2 + BE^2 = AB^2$ = the given sum of sqs. Also $\frac{AE}{BE} = \frac{AC}{CB}$ (V. 3.) = the given ratio $\therefore$ AE and EB are the reqd. lines.

Second method. Let a and b denote the reqd. lines, so that $\frac{a}{b} = \frac{AC}{CB}$. Draw CD perp. to AC, making CD = CB. Join AD.

$\frac{b}{a}=\frac{\mathsf{CB}}{\mathsf{CA}}$ $\therefore$ $\frac{a^2+b^2}{a^2}=\frac{\mathsf{CB}^2+\mathsf{CA}^2}{\mathsf{CA}^2}=\frac{\mathsf{CD}^2+\mathsf{CA}^2}{\mathsf{CA}^2}=\frac{\mathsf{DA}^2}{\mathsf{CA}^2}$ $\therefore$ $\frac{\mathsf{AB}^2}{a^2}=\frac{\mathsf{DA}^2}{\mathsf{CA}^2}$ $\therefore$ $\frac{\mathsf{DA}}{\mathsf{CA}}=\frac{\mathsf{AB}}{a}$ $\therefore$ a is a fourth proportional to DA, CA, AB, and this can be found by V. 23. A fourth proportional to AC, CB, a gives b.

28. Let a, b, c denote the reqd. lines. Let AD be a str. line such that $a^2+b^2+c^2=\mathsf{AD}^2$. Divide AD at B and C in the given ratios, so that $a:b:c::\mathsf{AB}:\mathsf{BC}:\mathsf{CD}$. Draw BE perp. to BA and equal to BC. Join AE. Draw EF perp. to AE and equal to CD. Join AF. $\frac{b}{a}=\frac{\mathsf{BC}}{\mathsf{AB}}$ $\therefore$ $\frac{b^2+a^2}{a^2}=\frac{\mathsf{BC}^2+\mathsf{AB}^2}{\mathsf{AB}^2}=\frac{\mathsf{AE}^2}{\mathsf{AB}^2}$. Also $\frac{c}{a}=\frac{\mathsf{CD}}{\mathsf{AB}}$ $\therefore$ $\frac{c^2}{a^2}=\frac{\mathsf{EF}^2}{\mathsf{AB}^2}$ $\therefore$ $\frac{a^2+b^2+c^2}{a^2}=\frac{\mathsf{AE}^2+\mathsf{EF}^2}{\mathsf{AB}^2}=\frac{\mathsf{AF}^2}{\mathsf{AB}^2}$ $\therefore$ $\frac{\mathsf{AD}^2}{a^2}=\frac{\mathsf{AF}^2}{\mathsf{AB}^2}$ or $\frac{\mathsf{AF}}{\mathsf{AB}}=\frac{\mathsf{AD}}{a}$ $\therefore$ a is a fourth proportional to AF, AB, AD, and this is found as in V. 23. b and c can then be found by the method of the preceding example.

29. Let a and b denote the reqd. lines, a being the greater. Take AB such that $\mathsf{AB}^2=a^2-b^2=$ the given difference. Divide AB at C in the given ratio, so that $\frac{a}{b}=\frac{\mathsf{AC}}{\mathsf{CB}}$. On AC describe a semicircle ADC, and place a chord CD in it equal to CB. Join AD. $\frac{b}{a}=\frac{\mathsf{CB}}{\mathsf{CA}}$ $\therefore$ $\frac{a^2-b^2}{a^2}=\frac{\mathsf{CA}^2-\mathsf{CB}^2}{\mathsf{CA}^2}=\frac{\mathsf{CA}^2-\mathsf{CD}^2}{\mathsf{CA}^2}=\frac{\mathsf{AD}^2}{\mathsf{CA}^2}$ $\therefore$ $\frac{\mathsf{AB}^2}{a^2}=\frac{\mathsf{AD}^2}{\mathsf{CA}^2}$ $\frac{\mathsf{AD}}{\mathsf{CA}}=\frac{\mathsf{AB}}{a}$ $\therefore$ a fourth proportional to AD, CA, AB gives us the line a, and this can be found by V. 23. A fourth proportional to AC, CB, a gives us the line b.

30. Let ABC be the given $\triangle$. Draw BD perp. to BA and equal to it. With centre A and rad. AD describe a circle meeting AB produced at E. Draw EF $\parallel$ to BC to meet AC produced at F. $\triangle$s AEF, ABC are equiangular (I. 20.) and $\therefore$ similar (V. 5.) $\therefore$ $\frac{\triangle\mathsf{AEF}}{\triangle\mathsf{ABC}}=\frac{\mathsf{AE}^2}{\mathsf{AB}^2}=\frac{\mathsf{AD}^2}{\mathsf{AB}^2}=2$ $\therefore$ $\triangle$AEF is similar to $\triangle$ABC and twice its area.

31. Let ABC be the given △. Produce BA to D making AD = AB, and on BD describe an equilateral △BDE. Join AE. With centre A and rad. AE describe a circle cutting AB produced at F. Draw FG ∥ to BC to meet AC produced at G. AE is perp. to BD (I 7.) ∴ $EA^2 = EB^2 - BA^2 = 3BA^2$; $\frac{\triangle AFG}{\triangle ABC} = \frac{AF^2}{AB^2} = \frac{AE^2}{AB^2} = 3$. Also the △s are similar by construction ∴ AFG is the reqd. △.

32. Let ABC be the given △. Draw BD perp. to BA and equal to 2BA. With centre A and rad. AD, describe a circle cutting AB produced at E. Draw EF ∥ to BC to meet AC produced at F. $\frac{\triangle AEF}{\triangle ABC} = \frac{AE^2}{AB^2} = \frac{AD^2}{AB^2} = \frac{5AB^2}{AB^2}$ (II. 11.) $= 5$. Also the △s are similar by construction ∴ AEF is the reqd. △.

33. Let ABC be the given △. To AB and AC find a mean proportional X, and from AB and AC (produced if necessary) cut off AD = AE = X. Join DE. Rect. AB . AC = X^2 = rect AD . AE ∴ △s ADE, ABC are equal in area (V. 10.). Also ADE is isosceles, and is ∴ the reqd. △.

34. Join OA, and divide it at F, so that $\frac{OF}{FA}$ = the given ratio. Draw FD ∥ to AC to meet AB at D. Join OD and produce it to meet AC at E. $\frac{OD}{DE} = \frac{OF}{FA}$ (V. 2.) = the given ratio.

35. Describe the circum-circle, and from it cut off a segment ACB containing an angle equal to the given vertical ∠ (III. 25.). Bisect arc AEB, on the opp. side of AB to the pt. C, at E. Divide AB at D in the given ratio of the sides. Join ED and produce it to meet the circle at C. Join AC, CB. Arc AE = arc BE ∴ ∠ACE = ∠BCE ∴ $\frac{AC}{CB} = \frac{AD}{DB}$ = the given ratio. ∴ ACB is the △ reqd.

36. Let △AOB be gr. than △COD. Take a pt. F in OA such that OF is a fourth proportional to OB, OC, and OD. Then $\frac{OB}{OC} = \frac{OD}{OF}$ ∴ OB . OF = OC . OD ∴ △BOF = △COD (V. 10.). Thro. F draw FE ∥ to OB to meet AB at E. Join OE. △BEO = △BOF (II. 5.) = △COD ∴ BEO is the reqd. △.

37. Make AD = 1·5 in. so that DB = ·9 in. $\frac{AD}{DB} = \frac{15}{9} = \frac{5}{3}$. In AB produced make BE = 3·6 in. $\frac{BE}{EA} = \frac{3\cdot6}{6} = \frac{3}{5}$. Then as in Exercises lv. 3, we see that the circle on DE as diameter is the locus.

38. Let $\frac{AC}{CB}$ be the given ratio, ACB being a str. line. On AB describe a semicircle and draw CD perp. to AB to meet it at D. Join DA, DB. In CD make CE equal to a side of the given sq., and draw EF, EG ∥ to DA, DB respectively to meet AB in F and G. AC . CB = CD^2 (V. 9.). Also △s ECF, ACD are similar and △ECG, DCB ∴ $\frac{CE}{CF} = \frac{CD}{CA}$ and $\frac{CE}{CG} = \frac{CD}{CB}$ ∴ $\frac{CE^2}{CF.CG} = \frac{CD^2}{CA.CB}$ $= 1$ ∴ $CE^2 = CF.CG$. Also $\frac{CF}{CA} = \frac{CE}{CD}$ (V. 2.) $= \frac{CG}{CB}$ ∴ $\frac{CF}{CG} = \frac{CA}{CB}$ ∴ the rect. CF . CG in the reqd. rect.

39. Let AB be a side of the given equilateral △. Divide AB at C in the given ratio. On AB describe a semi-circle, and draw CD perp to AB to meet it at D. Join AD, BD. $\frac{\text{Equilateral } \triangle \text{ on } AD}{\text{Equilateral } \triangle \text{ on } BD} = \frac{AD^2}{BD^2} = \frac{AC.AB}{BC.BA} = \frac{AC}{BC}$. Also the sum of these △s = equilateral △ on AB (V. 15.) ∴ the equilateral △s on AD, BD are the reqd. △s.

40. On AB, 3·8 in. long, describe a semi-circle ABC. Draw BD, 1·3 in. long, at rt. ∠s to BA. Draw DC ∥ to BA to meet the semi-circle at C, and draw CE perp. to BA. ∠ACB is a rt. ∠ ∴ AE . EB = CE^2 (V. 9.) = BD^2 ∴ AE, EB are the reqd. segments. By measurement, AE = 3·28 in. and BE = ·52 in.

41. Let ABCDE be the given figure, *ab* (∥ to AB) the given str. line. Using the method of I. 31. (1st method), draw *bc* ∥ to BC, *ac* ∥ to AC, *cd* ∥ to CD, *ad* ∥ to AD, *de* ∥ to DE, *ae* ∥ to AE, *abcde* is similar to ABCDE (V. 25.).

42. Let D be the given pt. in AB, BD being gr. than DA. Take E in BC such that BE = $\frac{1}{3}$BC (V. 20.). Join DE and draw AF ∥ to DE to meet AB in F. Join DF. △DEF = △DAE ∴

$\triangle$DFB = $\triangle$AEB = $\frac{1}{3}\triangle$ABC. If CF > FB, from FC cut off FG = FB. $\triangle$DFG = $\triangle$DFB (II. 6.) = $\frac{1}{3}\triangle$ABC and the problem is solved. If CF < FB, as in the first part, find a point H in AC such that $\triangle$AHD = $\frac{1}{3}\triangle$ABC. The pt. F might also be found as follows. To BD, AB, BE find a fourth proportional BF (V. 23.). Then $\frac{BD}{AB} = \frac{BE}{BF}$ $\therefore$ BD . BF = AB . BE $\therefore$ $\triangle$BDF = $\triangle$ABE (V. 10.) = $\frac{1}{3}\triangle$ABC.

43. Let ABC be the $\triangle$, P the given pt. within it. Join PA, PB, PC. Draw AD $\parallel$ to PB to meet CB produced at D; and draw AE $\parallel$ to PC to meet BC produced at E. Join PD, PE. $\triangle$ABP = $\triangle$BPD and $\triangle$APC = $\triangle$PCE (II. 5.) $\therefore$ $\triangle$DPE = $\triangle$ABC. Divide DE into four equal parts, DL, LM, MN, NE (V. 21.). Let M fall within BC, L and N without it. Join PM. Draw LQ $\parallel$ to PB to meet AB at Q, and NR $\parallel$ to CP to meet AC at R. Join PQ, PR. $\triangle$AQP + $\triangle$BQP = $\triangle$APB = $\triangle$PBD (II. 5.) = $\triangle$PBL + $\triangle$PLD. But $\triangle$BQP = $\triangle$BPL (II. 5.) $\therefore$ $\triangle$AQP = $\triangle$PLD = $\frac{1}{4}\triangle$PDE = $\frac{1}{4}\triangle$ABC. In like manner, $\triangle$APR = $\frac{1}{4}\triangle$ABC. Also quadl. MPQB = $\triangle$PBM + $\triangle$PQB = $\triangle$PBM + $\triangle$PBL (II. 5.) = $\triangle$LMP = $\frac{1}{4}\triangle$PDE = $\frac{1}{4}\triangle$ABC. In like manner, quadl. PMCR = $\frac{1}{4}\triangle$ABC $\therefore$ the str. lines PQ, PM, PR divide the $\triangle$ into four equal parts.

44. Let ABC be the $\triangle$. Draw AD perp. to BC, and bisect BC at E, E falling between D and B. Let BGH be the reqd. $\triangle$, GH being perp. to BC at G. $\frac{2\triangle BGH}{\triangle ABC} = \frac{BG \,.\, GH}{\frac{1}{2}BC \,.\, AD} = \frac{BG \,.\, GH}{BE \,.\, AD} = \frac{BG^2}{BE \,.\, BD}$, for by similar $\triangle$s, $\frac{GH}{AD} = \frac{BG}{BD}$ $\therefore$ BG^2 = BE . BD. Hence the following construction. To BE and BD take a mean proportional BG along BC. Draw GH perp. to BC to meet AB at H. $\triangle$BGH = $\frac{1}{2}\triangle$ABC.

45. Let ABCD be the quadl., O the given pt. in AB. Join OD, OC. Draw AE $\parallel$ to OD to meet CD produced at E. Draw BF $\parallel$ to OC to meet DC produced at F. Join OE, OF. $\triangle$OEF = fig. ABCD (II. 5.). Bisect EF at G. Join OG. Fig. ADGO = $\triangle$OGD + $\triangle$OAD = $\triangle$OGD + $\triangle$OED (II. 5.) = $\triangle$EOG = $\frac{1}{2}\triangle$OEF = $\frac{1}{2}$ fig. ABCD $\therefore$ OG bisects the quadl.

46. Let O be the given pt. in the side AB of the quadl. ABCD. Reduce the quadl. to a triangle OEF as in the preceding example. Divide EF into three equal parts EG, GH, HF (V. 21.). Join OG, OH. As in the preceding, fig. DGOA $= \triangle$EOG $= \frac{1}{3}$ quadl. ABCD. In like manner $\triangle$OGH = fig. OHCB $= \frac{1}{3}$ quadl. ABCD $\therefore$ OG, OH divide the quadl. into three equal parts.

47. Let ABCD be the sq. Join AC, BD. Draw AE perp. to AC, and make $\angle$ACE equal to 30°. With centre A and rad. AE describe a circle cutting AD at F. Draw FG $\parallel$ to BD, to meet AB at G. EC = 2AE. $EC^2 - AE^2 = AC^2 \therefore 3AE^2 = AC^2 = 2AB^2 \therefore \frac{AE^2}{AB^2} = \frac{2}{3}$. Also $\frac{\triangle AFG}{\triangle ADB} = \frac{AF^2}{AD^2} = \frac{AE^2}{AB^2} = \frac{2}{3} \therefore \triangle AFG = \frac{2}{3}\triangle ADB = \frac{1}{3}$ sq. ABCD. Cutting off DH from DC equal to DF, and drawing HK $\parallel$ to DB, $\triangle$CHK $= \frac{1}{3}$ sq. ABCD $\therefore$ GF and HK, $\parallel$ to BD, trisect the sq.

48. In AB 5 cms. long take a pt. C such that $\frac{CB}{CA} = \frac{3}{5}$ (V. 21.). On AB describe a segment of a circle containing an angle of 60° and use the method of Example 35 above.

49. Draw $\angle$BAC = 45°, and take $\frac{BA}{AC} = \frac{3}{5}$. Join BC. Draw AD perp. to BC and from it, produced if necessary, cut off AE = 6·5 cms. Draw FEG perp to AE, meeting AC, AB at F and G. By similar $\triangle$s $\frac{AC}{AF} = \frac{AD}{AE} = \frac{AB}{AG} \therefore \frac{AF}{AG} = \frac{AC}{AB} = \frac{5}{3}$. Also the alt. of $\triangle$AFG = 6·5 cms. $\therefore$ AFG is the $\triangle$ reqd.

50. Draw AB = 3·7 cms. and from it produced cut off AC = 8·6 cms. Draw AE perp. to AB and equal to 1 cm. Take X a fourth proportional to AB, AE, and AC, and from AE produced cut off AF = X. Draw FHK $\parallel$ to AB, and bisect $\angle$CAF by AH meeting FK at H. Draw BK $\parallel$ to AH. ABKH is the reqd. parm. Draw BG perp. to AB to meet FHK at G. Parm. ABKH = rect. ABGF (II. 3.) = AB . AF = AB . X = AE . AC = 8·6 sq. cms.

51. Make $\angle$ACB = 41°, and cut off AC = 3·1 in. With centre A and rad. 2·7 in. describe a circle cutting BC at B. Make $\angle$BCD = 63° and let CD meet BA produced at D. To BA

and BD find a mean proportional X (V. 24.), and from BD cut off BE = X. Draw EF || to CD to meet CB at F. $\frac{\triangle EBF}{\triangle BCD} = \frac{BE^2}{BD^2} = \frac{BA \cdot BD}{BD^2} = \frac{BA}{BD} = \frac{\triangle ABC}{\triangle CDB}$ $\therefore$ $\triangle EBF = \triangle ABC$ and $\angle EFB = \angle DCB = 63°$ $\therefore$ EBF is the reqd. $\triangle$.

52. Draw perp. diameters AOB, COD $\therefore$ $\angle ACB = \angle$ in semicircle = a rt. $\angle$. Similarly $\angle$s at A, B, D are rt. $\angle$s. Also from $\triangle$s AOC, BOC, AC = BC (I. 4.) $\therefore$ ADBC is a sq. With centre D and rad. DO describe a circle cutting the first circle in E and F. C, E, F are alternate pts. of a regular hexagon in the circle. $\therefore$ CEF is an equilateral $\triangle$. To EF and AD take a third proportional X (V. 22.). $\frac{EF}{X} = \frac{EF^2}{AD^2}$ (V. 12. Cor. 2.) $= \frac{\text{sq. on side of equilateral } \triangle}{\text{sq. in the circle}}$. By measurement the ratio of X to EF will be found to be 2 : 3.

53. Let AB, BC be the given str. lines, O the given pt. Draw OD || to CB to meet BA at D. In BD produced make DA = 2BD. Join AO and produce it to meet BC at C. $\frac{AO}{OC} = \frac{AD}{DB}$ (V. 2.) $= \frac{2}{1}$ $\therefore$ AOC is the reqd. line.

54. Let ABCD be the given sq. Make $\angle BAE$ 60°, the pt. E being in BC produced. From BE cut off BF a mean proportional to BC and BE. Draw FG || to EA to meet AB at G. In AB produced make BH = BF. Join FH, AC. $\frac{\triangle GFB}{\triangle AEB} = \frac{BF^2}{BE^2} = \frac{BC}{BE}$ $= \frac{\triangle ABC}{\triangle AEB}$ (V. 1.) $\therefore$ $\triangle GFB = \triangle ABC$ $\therefore$ $\triangle GFH = 2\triangle ABC$ = square ABCD. Also $\triangle GFH$ is equilateral, since $\angle H = \angle BGF = 60°$.

55. Let ABCD be the sq. Take $BE = \frac{1}{3}BA$ (V. 20.). Make $\angle PBE = 60°$, P falling in CD. Join PE. $\triangle PBE = \frac{1}{6}$ sq. ABCD. To BE and BP take a mean proportional FG (V. 24.). On FG describe an equilateral $\triangle$FOG. $\triangle FOG = \triangle PBE$ (V. 10.). With centre O and rad. OF or OG describe a circle, and round its circumference set off chords GH, HK, KL, LM each equal to its radius. FGHKLM is a regular hexagon and is equal to six times $\triangle$FOG, and is therefore equal to the sq.

56. Let ABC be the equilateral triangle. Bisect AC at D. From AC cut off AF a mean proportional to AD, AC, and draw

FG $\parallel$ to CB to meet AB in G. Draw FH, GH perp. to AF, AG; FK, AK perp. to FG, AG; AL, GL perp. to FA, FG. The hexagon ALGHFK $= 2\triangle$AFG $= \triangle$ABC, for $\frac{\triangle AFG}{\triangle ABC} = \frac{AF^2}{AC^2} = \frac{AC \cdot AD}{AC^2} = \frac{AD}{AC} = \frac{1}{2}$.

57. Let O be the pt. within the circle. Draw any chd EOF and take OD a mean proportional to OE and OF (V. 24.). If x and y are the segments of the reqd. chord, $x \cdot y =$ OE . OF $=$ OD^2 ... (1). Take a line p such that $\frac{p}{OD} =$ the given ratio $= \frac{x}{y}$ ... (2). From (1) $\frac{x}{OD} = \frac{OD}{y} = \frac{p}{x}$ $\therefore$ $x^2 = p$. OD ... (3). Hence the following construction. Take p (as above) such that the given ratio $= \frac{p}{OD}$ (V. 23.) then one segment [by (3)] is a mean proportional between p and OD.

58. On AB the given base describe a segment of a circle containing an angle equal to the given vertical angle. Draw AC a diameter. Let P, Q be the sides of the given rectangle. To AC, P and Q take a fourth proportional X. At A draw AD perp. to AB and equal to X. Draw DE $\parallel$ to AB to meet the circle at E. Draw EF perp. to AB. Rect. AE . EB $=$ rect. AC . EF (V. 18.) $=$ AC . X $=$ P . Q. Also $\angle$AEB $=$ the given angle $\therefore$ AEF is the reqd. $\triangle$.

59. Draw AB 3 in. long, and with centre A and rad. 2·3 in. describe a circle. Draw BC perp. to AB $\frac{2}{3}$ in. long, and draw CD $\parallel$ to BA to meet the circle at D. Join DB. Draw DE perp. to AB. Area of $\triangle$ADB $= \frac{1}{2}$AB . DE $= \frac{1}{2} \times 3 \times \frac{2}{3} = 1$ sq. in. By measurement DB $=$ 1·04 in. If we take the other pt. where CD meets the circle, we obtain a second $\triangle$ satisfying the given conditions.

60. Draw a str. line ABCD, such that AB $=$ 1 in., BC $=$ 3 in., BD $=$ 4·7 in. On AC and AD (on the same side of the line) draw semi-circles. Draw BEF perp. to ABC to meet the circles at E and F. Join EA, EC. Draw FG $\parallel$ to EA and FH $\parallel$ to EC to meet ABC at G and H. $\frac{BG}{BA} = \frac{BF}{BE} = \frac{BH}{BC}$ $\therefore$ $\frac{BG}{BH} = \frac{BA}{BC} = \frac{1}{3}$. Also BG . BH $=$ BF^2 (for $\angle$GFH $= \angle$AEC $=$ a rt. $\angle$) $=$ BA . BD $=$ 4·7 sq. in. $\therefore$ the rect. contained by BG and BH is the reqd. rect.

61. Draw a str. line AB 3·28 in. long, and AC at rt. $\angle$s to it 2 in. long. Join BC. Make $\angle$BAD = 60°, and draw CD ∥ to AB. To AB and AD find a mean proportional X, and from AD and AB cut off AF = AE = X. Join EF. $\angle$FAE = 60° and $\angle$AFE = $\angle$AEF $\therefore$ $\triangle$AEF is equilateral. Also $\dfrac{\triangle AEF}{\triangle ADB} = \dfrac{AE.AF}{AD.AB}$ (V. 10.) $\therefore$ $\triangle AEF = \triangle ADB = \triangle ACB = \frac{1}{2} \times 2 \times 3{\cdot}28 = 3{\cdot}28$ sq. in.

62. Draw the diagonals AOC, BOD of the quadl. ABCD. On OA describe an equilateral $\triangle$OEA, and draw EF perp. to OA. From OA cut off OG = EF. Draw GH ∥ to AD to meet OD at H, HK ∥ to DC to meet OC at K, KL ∥ to CB to meet OB at L, and join LG. $\dfrac{\triangle OGH}{\triangle ODA} = \dfrac{OG^2}{OA^2} = \dfrac{EF^2}{OA^2} = \dfrac{EF^2}{OE^2} = \dfrac{3}{4}$ for $OF = \frac{1}{2}OA = \frac{1}{2}OE$. Also since $\dfrac{OG}{OA} = \dfrac{OH}{OD} = \dfrac{OK}{OC} = \dfrac{OL}{OB}$, $\triangle$s OHK, OCD, $\triangle$s OLK, OCB, and $\triangle$s OLG, OBA are in the same ratio. Also these pairs of $\triangle$s are similar $\therefore$ the quadls. FHKL, ABCD are similar $\therefore$ FHKL is the reqd. quadl.

63. Describe the circum-circle. Let AE be a diameter of it. Make $\angle$EAF equal to the given difference of angles, and make AF equal to the side AD of a rectangle on AE equal to the given rectangle contd. by the sides. Through F draw a perp. to AF to meet the circumference in B, C. Join AB, AC. AB . AC = AE . AF (V. 18.) = the given rectangle. $\angle C - \angle B = 90° - B - (90° - C) = \angle BAF - \angle ECB = \angle BAF - \angle EAB$ (III. 12.) = $\angle$EAF = the given angle.

64. If BAC is the reqd. $\triangle$ on given base BC, AB being gr. than AC, let the bisector of the ext. $\angle$CAE meet BC produced at O. $\dfrac{BO}{OC} = \dfrac{BA}{AC}$ = the given ratio. Also $\angle AOC = C - \frac{1}{2}\angle CAE = C - \frac{1}{2}(B + C) = \dfrac{C - B}{2} = \frac{1}{2}$ the given diff. of base angles. Hence the following construction. Let $\dfrac{a}{b}$ = the given ratio, $a > b$. Take a line X such that $\dfrac{X}{BC} = \dfrac{b}{a-b}$ and in BC produced make CO = X. $\dfrac{CO}{BC} = \dfrac{b}{a-b}$ $\therefore$ $\dfrac{CO}{BO} = \dfrac{b}{a}$. Thro. O draw OA making $\angle$BOA

$=\frac{1}{2}$ given diff. of base angles. Draw CD perp. to OA and produce it to E, making DE = DC. Join BE cutting ODA at A. Join AC. In $\triangle$s ADC, ADE, $\angle$CAD = $\angle$DAE (I. 4.) $\therefore \frac{BA}{AC}=\frac{BO}{OC}$ (V. 4.) $=\frac{a}{b}$. Also $\angle C - \angle B = 2\angle AOC$ = given $\angle$, as above; $\therefore$ ABC is the $\triangle$ reqd.

EXERCISES LXV.

1. $\triangle$s ABE, BCF are equal in all respects (I. 4.) $\therefore \angle BEG = \angle BFC$ $\therefore$ $\triangle$s BEG, BFC are equiangular $\therefore \frac{GE}{GB}=\frac{CF}{BC}=\frac{1}{2}$. Also $\triangle$s BGE, AGB are equiangular $\therefore \frac{AG}{BG}=\frac{BG}{GE}=2$ $\therefore$ AG = 2BG = 4GE $\therefore$ GE $=\frac{1}{5}$AE. From $\triangle$s AHB, CHF, $\frac{BH}{HF}=\frac{AB}{FC}=2$ $\therefore$ BH = 2HF $\therefore$ BH $=\frac{2}{3}$BF $=\frac{2}{3}$AE $=\frac{10}{3}$GE, *i.e.* BG + GH $=\frac{10}{3}$GE $\therefore$ GH $=\frac{10}{3}$GE − BG $=\frac{10}{3}$GE − 2GE $=\frac{4}{3}$GE, or GE $=\frac{3}{4}$GH.

2. $\frac{BD}{EC}=\frac{\triangle ABD}{\triangle EAC}$ (V. 1.) $=\frac{BA \,.\, AD}{AC \,.\, AE}$ (V. 10.) $\therefore \frac{BA}{AC}=\frac{AE \,.\, BD}{AD \,.\, EC}$ ... (1). Also $\frac{BE}{CD}=\frac{\triangle ABE}{\triangle ACD}$ (V. 1.) $=\frac{BA \,.\, AE}{CA \,.\, AD}$ $\therefore \frac{BA}{AC}=\frac{BE \,.\, AD}{AE \,.\, CD}$ $\therefore$ from (1) $\frac{BA^2}{AC^2}=\frac{BD \,.\, BE}{CD \,.\, CE}$.

3. Let the line joining A to the mid. pt. of the base BC meet CD at O. AO bisects $\angle$BAC (I. 7.) $\therefore \frac{DO}{CO}=\frac{DA}{CA}$ (V. 3.) $=\frac{DA}{AB}=\frac{BD}{AD}$, for AB . BD = AD^2 $\therefore$ CD is divided in the same ratio as AB, *i.e.* it is divided in extreme and mean ratio.

The solutions of **4**–**8** will be found on page 178.

EXERCISES LXVI.

1. and **2.** Done in the text-book.

3. Let A and B be the fixed pts. and P one position of the moving pt. Let the internal and external bisectors of the $\angle$P meet AB and AB produced at C and D. $\frac{AC}{CB}=\frac{AP}{PB}$ (V. 3.) = a

constant ∴ C is a fixed point for all positions of P. Similarly by V. 4. D is a fixed pt. ∴ the locus of P is a circle on CD as diameter, for ∠CPD is a rt. ∠.

4. Let AP be the given str. line. Draw OA perp. to AP, and divide it at B so that $\frac{OB}{BA}=\frac{2}{1}$. Draw OP, and divide it at Q so that $\frac{OQ}{QP}=\frac{2}{1}$. Join BQ. Q is a pt. on the locus. $\frac{OQ}{QP}=\frac{OB}{BA}$ ∴ QB is ∥ to AP ∴ ∠OBQ is a rt. ∠. Also B is a fixed pt. ∴ the str. line QB is the locus reqd., viz. a str. line ∥ to the given str. line.

5. Draw the str. line AB to meet the given ∥ str. lines, CD, EF at B and A. Bisect AB at P, and draw MPN perp. to CD and EF. From △s APN, BPM, PN = PM, and MN is constant ∴ the locus of P is a str. line ∥ to the given str. lines and equidistant from them.

6. If AB be the given base, P any position of the vertex on the given line PE, Q the intersn. of the medians lies in DP such that $\frac{DQ}{QP}=\frac{1}{2}$, when D is the mid. pt. of AB. Thus we see, as in Example 4 above, that the locus of Q is a str. line ∥ to the given str. line.

7. On the given base BC describe a segment, BAC, of a circle containing an angle equal to the given vertical angle (III. 23.). Then the arc is the locus of the vertex of the △ (III. 12.). Bisect BC at D. Join AD. Then the intersn. of the medians lies at P in DA, where $\frac{DP}{PA}=\frac{1}{2}$. Draw PE ∥ to AB, and PF ∥ to AC to meet BC at E and F. $\frac{DE}{EB}=\frac{DP}{PA}=\frac{2}{1}$ ∴ E is a fixed pt. In like manner, F is a fixed pt. Also ∠EPF = ∠BAC (I. 20.) = the given ∠ ∴ the locus of P is an arc of a circle on EF, containing an angle equal to the given angle.

8. The base and area being constant, the altitude of the △ is constant, *i e.* the vertex lies on a fixed str. line ∥ to the base. The locus can now be found as in Example 4 above.

9. (1) When the given str. lines are ||, the locus is a str. line || to them.

(2) When they intersect, let AB, AC be the given lines. Let P be any pt. on the locus, and draw PM, PN perp. to AB and AC. In AP take any pt. Q and draw QR, QS perp. to AB and AC. By similar △s $\frac{QS}{PN} = \frac{QA}{PA} = \frac{QR}{PM}$ ∴ $\frac{QS}{QR} = \frac{PN}{PM}$ ∴ Q is a pt. on the locus ∴ the locus is the str. line AP. The line AP is however only part of the locus, for in the same way it may be shown that pts. on a str. line thro. A and within the ∠ formed by AB and AC produced satisfy the given condition.

10. Describe any circle PAD passing thro. the pts. A and D. Also thro. P, B, C, describe another circle cutting the former circle again at Q. Join QP and produce it to meet AD at O. Rect. OA . OD = rect. OP . OQ = rect. OB . OC (IV. 14.). To OA and OD take a mean proportional OE in the line OABCD. With centre O and rad. OE describe a circle ERS and take any pt. R on it. $OA \cdot OD = OE^2 = OR^2$ ∴ $\frac{OA}{OR} = \frac{OR}{OD}$ ∴ △s OAR, ORD are similar ∴ ∠ORA = ∠ODR. Also since $OB \cdot OC = OA \cdot OD = OE^2$ we can prove in the same way that ∠ORB = ∠OCR. But ∠ARB = ∠ORB − ∠ORA and ∠CRD = ∠OCR − ∠ODR ∴ ∠ARB = ∠CRD ∴ the locus of pts. at which AB and CD subtend equal ∠s is the circle ERS. By lxv. 2. it can be shewn to be the same question as lxvi. 3.

11. Let ABC be the △; P, Q, R the given str. lines. It is reqd. to find a pt. O such that the perps. from O on the sides BC, CA, AB are proportional to P, Q, R. On the same side of BC as A, draw BD perp. to BC and equal to P. On the same side of AC as B draw CE perp. to AC and equal to Q. Draw DF and EF || to BC and AC, meeting at F. The perps. from F on BC and AC are equal to P and Q respectively ∴ the pt. reqd. lies in CF. In the same way a line BG may be found in which the reqd. pt. lies ∴ it lies at the pt. where CF and BG meet.

12. Let A be the fixed angular pt., B any position of the angular pt. in the given line EF, and C the vertex whose locus it is reqd. to find. If the circum-circle of △ABC meet EF at G,

$\angle AGB$ = supplement of $\angle C$ and is $\therefore$ constant. Also A is a fixed pt. $\therefore$ AG is a fixed line. Join GC. $\angle BGC = \angle BAC$ in the same segment $\therefore$ $\angle BGC$ is fixed $\therefore$ the locus of C is the fixed str. line GC.

EXERCISES LXVII.

1. Let AB be the polar of P, CD the polar of Q. Let AB, CD meet at O. The polar of P passes thro. O $\therefore$ the polar of O passes thro P. In like manner the polar of O passes thro. Q $\therefore$ PQ is the polar of O.

2. With the fig. of Example 1 above, O is the pole of PQ.

3. With the fig on p. 372, let P be the fixed pt. Q is the pole of the str. line ABP. It is reqd. to find the locus of Q. This is proved on p. 372 to be a str. line at rt. $\angle$s to OP.

4. If P and Q be the pts. and O the centre of the circle, OP and OQ are respectively perp. to the polars of P and Q $\therefore$ the $\angle POQ$ = the $\angle$ between the polars of P and Q.

EXERCISES LXVIII.

1. Let PB, P′B′ be two || radii of the circles (see fig. on p. 376). $\angle OBP = \angle OB'P'$ and $\angle O$ is common to $\triangle$s OPB, OB′P′ $\therefore \dfrac{OP}{OP'} = \dfrac{PB}{P'B'}$, *i.e.* PP′ is divided externally in the ratio of the radii $\therefore$ O is a centre of similitude. In like manner, if PB, P′B′ are drawn in opp. directions, it will be seen that BB′ divides PP′ internally in the ratio of the radii, and $\therefore$ passes thro. a centre of similitude.

2. This is best proved by means of the prop. on Transversals on p. 379. Let A, B, C be the centres of the circles R_1, R_2, R_3 their radii. Let P be the internal c. of similitude of A and B, Q the internal c. of similitude of C and A, R the external c. of similitude of B and C. $\dfrac{AP}{PB} = \dfrac{R_1}{R_2}$, $\dfrac{CQ}{AQ} = \dfrac{R_3}{R_1}$, and $\dfrac{BR}{CR} = \dfrac{R_2}{R_3}$ $\therefore$ $\dfrac{AP \cdot BR \cdot CQ}{BP \cdot CR \cdot AQ} = 1$ $\therefore$ P, Q, R are collinear (prop. on p. 379). In like manner, the rest of the prop. may be proved.

3. Let a circle (centre A) touch the circles whose centres are C and D at P and W. Also let a circle (centre B) touch the circles whose centres are C and D at Q and V. Take the case where P and W lie on one side of CD and Q and V on the opp. side, all the contacts being external. Let QP meet circle A again at R. Join CA (passing thro. P), CB (passing thro. Q), and AR. From the isos. $\triangle$s CPQ, ARP we see that CQ is $\parallel$ to AR, *i.e.* QB and AR are $\parallel$ $\therefore$ QR passes thro. the external c. of similitude of circles A and B. In like manner, VW passes thro. the same c. of similitude. Let O be this centre. Draw OEF a common tangent to circles A and B. $OP \cdot OQ = OE \cdot OF = OW \cdot OV$ (p. 376) $\therefore$ O is a pt. on the radical axis of circles C and D. In the same way we may prove the other cases of the prop.

4. Let A and B be the centres of the fixed circles. Take the case where the variable circle of centre P touches them both externally. Join PA, PB passing thro. the pts. of contact, C, D. Let DC meet the circle (centre A) again at E. Join AE. From the isos. $\triangle$s PCD, ACE we see that PD is $\parallel$ to AE, *i.e.* BD is $\parallel$ to AE $\therefore$ DE passes thro. the external c. of similitude of the fixed circles. In the same way the other cases may be proved.

EXERCISES LXIX.

1. Take ABC so that AB and AC are equal to the given str. lines. On BC as diameter describe a circle, and from A draw tangents AD, AE to it. If the chd. of contact DE meets BC at F, AF will be the H.M. reqd. Join BD, DC. $\angle ADB = \angle BCD$ (in alt. segt.) $= \angle BDF$ (V. 9.) $\therefore$ DB bisects the intr. $\angle ADF$ of $\triangle ADF$. BDC is a rt. $\angle$ $\therefore$ DC bisects the extr. $\angle$ of $\triangle ADF$ $\therefore \dfrac{AC}{CF} = \dfrac{AB}{BF}$ $\therefore$ AC, AF, AB are in H.M. (p. 384).

2. With the same figure $AB + AC = 2AB + BC = 2AB + 2BO = 2AO$ $\therefore$ AO is the A.M. between AB and AC. Also $AD^2 = AB \cdot AC$ (IV. 14.) $\therefore$ AD is the G.M. between AB and AC. From similar $\triangle$s ADF, AOD $\dfrac{AO}{AD} = \dfrac{AD}{AF}$, *i.e.* the A.M., G.M., and H.M. are in continued proportion.

3. From an external pt. O draw OA, OB, tangents to, and OPRQ cutting at P, Q the circle whose centre is D. Join AB cutting OD at C and OQ at R. $OP \cdot OQ = OA^2$ (IV. 14.) $= OC \cdot OD$ for $\angle ACO$ is a rt. $\angle$ $\therefore$ P, Q, D, C are concyclic $\therefore$ $\angle PCO$ = supplement of $\angle PCD = \angle PQD = \angle DPQ = \angle DCQ$ $\therefore$ CO bisects the extr. $\angle$ of $\triangle PCQ$. Also ACO is a rt. $\angle$ $\therefore$ CA bisects the intr. $\angle PCQ$ $\therefore$ $\frac{OQ}{OP} = \frac{CQ}{CP}$ (V. 4.) $= \frac{QR}{RP}$, *i.e.* PQ is divided internally and externally in the same ratio at R and O $\therefore$ OQ is cut harmonically at P and R.

4. Let the transversal DPEQ meet BC at P, CE at E, and AC produced at Q. From the similar $\triangle$s QEC, QDA, $\frac{QD}{QE} = \frac{AD}{CE} = \frac{DB}{CE} = \frac{DP}{PE}$ from the similar $\triangle$s BPD, CPE $\therefore$ DE is divided internally at P, and externally at Q in the same ratio $\therefore$ the pts. D and E are harmonically conjugate with respect to the dts. P and Q.

5. From the similar $\triangle$s AEC, AOD, $\frac{EC}{OD} = \frac{AC}{AD} = \frac{BC}{BD}$ (hyp.) $= \frac{CF}{OD}$ from the similar $\triangle$s CBF, DBO $\therefore$ CE = CF.

6. Draw A′C′B′D′ any transversal cutting OA, OC, OB, OD at A′, C′, B′, D′. Draw E′C′F′ ‖ to OD cutting OA, OB at E′ and F′. As in Example 5 above, E′C′ = C′F′. Thus in $\triangle$OE′F′ the transversal A′C′B′D′ is drawn through C′ the mid. pt. of E′F′ $\therefore$ as in Example 4 above, A′B′ is divided harmonically at C′ and D′.

7. To CA and CB take a mean proportional X, and from CBA cut off CE = CF = X. $CA \cdot CB = CE^2$ $\therefore$ $\frac{CA}{CE} = \frac{CE}{CB}$ $\therefore$ $\frac{CA + CE}{CA - CE} = \frac{CE + CB}{CE - CB}$, *i.e.* $\frac{AF}{AE} = \frac{BF}{BE}$ $\therefore$ AB is divided harmonically at E and F in such a manner that CE = CF.

8. Let OC and OD bisect internally and externally the $\angle AOB$. Draw any transversal ACBD. $\frac{AD}{DB} = \frac{AO}{OB}$ (V. 4.) $= \frac{AC}{CB}$ $\therefore$ AB is divided harmonically at C and D.

EXERCISES LXX.

1. Let A be the fixed pt., P one of the pts. in the plane, AB perp. to the plane. $BP^2 = AP^2 - AB^2$ (II. 11.) = a constant, since AP is of constant length; ∴ the locus of P is a circle whose centre is the fixed pt. B.

2. Let AB be the fixed str. line, P any position of the moving pt. If P move only in the plane PAB its locus is a str. line PQ ∥ to AB ∴ the locus of all the positions of P is the surface generated by allowing PQ to move without altering its distance from AB, *i.e.* a right cylinder whose axis is AB (VII. Def. 14.).

3. Let A, B be the given pts., P a position of the moving pt. If P remain in the plane PAB its locus is the line bisecting AB at rt. ∠s (I. 23.). The complete locus is found by revolving the figure about AB, and is therefore the plane bisecting AB at rt. ∠s.

4. Let A, B be the fixed pts., P the moving pt. Draw PN perp. to AB. $AN^2 - BN^2 = AP^2 - BP^2$ (II. 11.) = a constant ∴ N is a fixed pt. ∴ the locus of P is a plane cutting AB at rt. ∠s.

5. Let A, B be the fixed pts., P the moving pt., C the mid. pt. of AB $2CP^2 + 2AC^2 = AP^2 + BP^2$ = a constant ∴ CP = a constant ∴ the locus of P is a sphere whose centre is C.

6. Let A, B, C be the three given points, P the moving pt., PD perp. to the plane ABC. $PA^2 = PB^2 = PC^2$ (hyp.) ∴ $AD^2 = BD^2 = CD^2$ (II. 11.) ∴ D is the circum-centre of the △ABC ∴ the locus of P is the perp. to the plane ABC drawn through the circum-centre. There is no such point if A, B, C are collinear.

7. Cut off equal lengths OA, OB, OC. The reqd. line is the perp. OP from O to the plane ABC. For the right-angled △s OAP, OBP, OCP are equal in all respects (I. 17.) ∴ OP is equally inclined to OA, OB, OC.

8. Let ABDC be a skew quadl., E, F, G, H the mid. pts of AB, AC, DB, DC. EF is ∥ to BC and equal to $\frac{1}{2}$BC (Ex. xx. 1.); so also is GH. ∴ EF is equal and ∥ to GH. ∴ EFHG is a parm. (II. 1.).

9. As in the last question EF, GH are ∥ ∴ they are in one plane ∴ EH and GF are concurrent.

EXERCISES LXXI.

1. Let A, B be two points, E the mid. pt. of AB; C, F, D their projections on the plane. Let GEH be || to CD meeting CA, DB at G, H. $\triangle$GAE = $\triangle$HBE in all respects (I. 16.) $\therefore$ GA = BH $\therefore$ AC + BD = GC + HD = 2EF (II. 2.).

2. Let A, B, C be three pts., G their centroid, D the mid. pt. of BC. Let H, K, L, N, M be the projections of these 5 pts. on the plane. Let PGQ be || to HM meeting HA, MD in P, Q. From similar $\triangle$s PGA, QGD, $\frac{PA}{DQ} = \frac{AG}{GD} = \frac{2}{1}$. As in Question 1, BK + CL = 2DM = 2DQ + 2QM = PA + 2GN $\therefore$ AH + BK + CL = PH + 2GN = 3GN.

3. AP is perp. to the first plane, $\therefore$ the plane APQ which contains AP is perp. to the first plane (VI. 16.). Similarly the plane APQ is perp. to the second plane, $\therefore$ the plane APQ is perp. to the common section of the other two planes (VI. 17.).

4. Let PQR, SQR be intersecting planes. Let ABC, DEF be || planes meeting PQR in AB, DE, and SQR in BC, EF. AB is || to DE, and BC is || to EF (VI. 14.) $\therefore$ $\angle$ABC = $\angle$DEF (VI. 10.).

5. Let a str. line PQ meet a plane XY in Q Let QR be the projection of PQ. PQR is the minimum $\angle$. For in XY draw any other line QS making it equal to QR. PR is perp. to the plane $\therefore$ PRS is a rt. $\angle$ $\therefore$ PS > PR $\therefore$ in $\triangle$s PQS, PQR, $\angle$PQS is gr. than $\angle$PQR (I. 15.).

6. Let PO be equally inclined to three str. lines OA, OB, OC which are in a plane. Let PQ be perp. to the plane, Q lying within the $\angle$AOB. Make OA equal to OB. In $\triangle$s POA, POB, PA = PB (I. 4.) $\therefore$ in the right-angled $\triangle$s PQA, PQB, QA = QB (I. 17.) $\therefore$ in $\triangle$s AOQ, BOQ, $\angle$AOQ = $\angle$BOQ (I. 7.) $\therefore$ Q lies on the bisector of the $\angle$AOB. Similarly Q lies on the bisector of the $\angle$AOC $\therefore$ Q lies at the common pt., viz. O, *i.e.* PO is perp. to the plane.

7. Let AB, CD be || str. lines, EF, GH their projections on a plane XY. Let AK, CL be || to EF, GH. AE is || to CG (VI. 6.), AB is || to CD (hyp.) $\therefore$ the planes BAE, DCG are || (VI. 13.) $\therefore$ their intersections with XY, viz EF, GH are || (VI. 14.). The

$\triangle$s BAK, DCL are equiangular to each other (VI. 10.) $\therefore$ $\frac{AB}{CD} = \frac{AK}{CL} = \frac{EF}{GH}$.

8. Let ABCD be the base, O the vertex, OE the vertical height. $OA^2 = OE^2 + AE^2 = 2AE^2$, similarly $OB^2 = 2AE^2$, $AB^2 = AE^2 + EB^2 = 2AE^2$ $\therefore$ $OA = OB = AB$.

EXERCISES LXXII.

1. Let AB be the line L. BC its projection on Q. Let BD be the trace of Q on P, and let AG, CF be $\parallel$ to BD. AC is perp. to the plane Q $\therefore$ ACF is a rt. $\angle$ $\therefore$ CAG is a rt. $\angle$ (I. 20.). AB is perp. to the plane P $\therefore$ ABD is a rt. $\angle$ $\therefore$ BAG is a rt. $\angle$ (I. 20.) $\therefore$ AG is perp. to the plane ABC (VI. 4.) $\therefore$ DB is perp. to the plane ABC (VI. 8.) $\therefore$ DBC is a rt. $\angle$.

2. Let AB be $\parallel$ to a str. line CD, which is in a plane P. AB and CD are in one plane $\therefore$ if AB met the plane P in a point R, R would lie in the plane ABDC $\therefore$ R would lie in the intersection of the planes, *i.e.* in CD $\therefore$ AB and CD would not be parallel.

3. Let the planes be P, Q, R. If the lines of intersection are not $\parallel$, let AB, AC be the intersections of P with Q, and P with R. Then the pt. A lies on R $\therefore$ the intersection of Q with R passes through A.

4. Let A be the given pt., BC the given str. line, P a plane containing BC; AE perp. to the plane P. Let AD be perp. to BC. Join DE. D and A are fixed points, and AED a rt. $\angle$ $\therefore$ the locus of E is a circle whose diameter is AD; for E must lie in a plane through A perp. to BC (Question 1).

5. Let A be the centre. Join OA, and draw QB $\parallel$ to PA meeting OA at B. Then $\frac{OB}{BA} = \frac{OQ}{QP}$ = a given ratio $\therefore$ B is a fixed pt. Also $\frac{QB}{PA} = \frac{OQ}{OP}$ = a given ratio $\therefore$ QB = a constant $\therefore$ as P traces a circle round A, Q traces a circle round B, QB and PA remaining parallel.

6. The same as 4.

7. Let AB, CD be $\parallel$ str. lines; let BE, DF be their projections. AE is $\parallel$ to CF (VI. 6.), AB is $\parallel$ to CD (hyp.) $\therefore$ plane BAE is $\parallel$ to plane DCF (VI. 13.) $\therefore$ BE is $\parallel$ to DF (VI. 14.) $\therefore$ $\angle$ABE = $\angle$CDF (VI. 10.).

8. Let P be any position of the pt., PQ perp. to BC the intersection of the planes. Let R, S be the projections of P on the planes. PR = PS (hyp.) $\therefore$ PQ bisects $\angle$SQR (I. 17.) $\therefore$ the locus of P is a plane bisecting the angle between the given planes.

9. The locus is the str. line which is the intersection of planes bisecting the $\angle$s between two pairs of the given planes (see the preceding).

10. If the 4 planes form a tetrahedron, the pts. are the centres of spheres touching the 4 faces. A sphere may touch all internally, or one externally, the rest internally. 5 points.

EXERCISES LXXIII.

1. Let ABCD be a skew quadl., BD a diagonal. $\angle$ABC is less than $\angle$ABD + $\angle$CBD (VI. 18.). $\angle$CDA is less than $\angle$BDA + $\angle$BDC (VI. 18.) $\therefore$ $\angle$ABC + $\angle$BCD + $\angle$CDA + $\angle$DAB is less than the sum of the $\angle$s of the $\triangle$s ABD, CBD, *i.e.* less than 4 rt. $\angle$s.

2. Let BAC, CAD, DAB be the containing $\angle$s, AP any line. $\angle$PAB + $\angle$PAC > $\angle$BAC (VI. 18.). $\angle$PAC + $\angle$PAD > $\angle$CAD, $\angle$PAD + $\angle$PAB > $\angle$DAB $\therefore$ 2($\angle$PAB + $\angle$PAC + $\angle$PAD) > $\angle$BAC + $\angle$CAD + $\angle$DAB.

3. Let ABC be a $\triangle$, DEF an inscribed $\triangle$, P any pt. outside the plane of ABC. $\angle$BPF + $\angle$BPD > $\angle$FPD (VI. 18.). $\angle$CPD + $\angle$CPE > $\angle$DPE, $\angle$EPA + $\angle$APF > $\angle$EPF $\therefore$ by addition the result follows.

4. Let a plane BCD cut AO at rt. $\angle$s in D. Let the plane bisecting the dihedral $\angle$ (*i.e.* the angle BDC) meet BC at E. BO = CO, and BD = CD (I. 16.) $\therefore$ in $\triangle$s BDE, CDE, BE = CE (I. 4.) $\therefore$ in $\triangle$s BOE, COE, $\angle$BEO = $\angle$CEO (I. 7.), and the $\angle$s at E are rt. $\angle$s. $\therefore$ CE is perp. to EO and ED. $\therefore$ the plane CBO which contains CE is perp. to the plane DEO (VI. 16.).

5. Let AB, AC be equally inclined to a plane BCD. Let D be the projection of A. In $\triangle$s ABD, ACD, $\angle$ABD = $\angle$ACD (hyp.),

the angles at D are equal, and AD is common $\therefore$ AB = AC (I. 16.) $\therefore$ $\angle$ABC = $\angle$ACB (I. 5.).

6. Let AOE, BOF be the perpendiculars. The plane AOB contains AE and is therefore perp. to plane BCD (VI. 16.); it also contains BF and therefore is perp. to plane ACD $\therefore$ plane AOB is perp. to the line CD (VI. 17.).

7. Let O be the pt., AB, CD, EF be $\parallel$ str. lines. Let OP be a str. line $\parallel$ to AB. Then OP must be $\parallel$ to CD, EF etc. ... OP is $\parallel$ to AB $\therefore$ the plane OAB passes through P. Similarly the plane OCD passes through P, and so on $\therefore$ the planes have a common line OP $\therefore$ if any plane cuts these planes, the lines of intersection all pass through some point on OP.

8. Let ABC be a $\triangle$ right-angled at C, P a pt. equidistant from A, B, C; D the mid. pt. of AB. AD = BD = CD (Ex. xviii. 9.) $\therefore$ $\triangle$s PDA, PDB, PDC are equal in all respects (I. 7.) $\therefore$ $\angle$PDC = $\angle$PDA = $\angle$PDB = 90° $\therefore$ PD is perp. to plane ABC (VI. 4.).

9. In the figure of VI. 20. draw LP $\parallel$ to AB. Then LK is perp. to the plane DLP (VI. 4.) and to the plane HKB $\therefore$ these planes are parallel, and one contains CD, the other AB.

10. Let E. F, G, H be the mid pts. of AB, CD, CB, AD. FG, HE are $\parallel$ to BD and each = $\frac{1}{2}$BD (Ex. xx. 1.) $\therefore$ FEGH is a parm. $\therefore$ FE is in the same plane as EG, EH. GE is $\parallel$ to CA and therefore perp. to AB. Similarly EH is perp. to AB $\therefore$ AB is perp. to the plane HEG (VI. 4.) $\therefore$ AB is perp. to EF.

11. Let AHD, BHE be perp. to BC, CA; HP perp. to the plane ABC. The plane PBE is perp. to plane ABC (VI. 16.), and AC is drawn in the plane ABC perp. to the common section $\therefore$ AC is perp. to the plane PBE. But it follows from VI. 8. that if a str. line is perp. to a plane it is perp. to any str. line in that plane $\therefore$ AC is perp. to BP.

[NOTE.—In the figure of VI. 8. AB is $\parallel$ to CD, and CD is perp. to EF $\therefore$ AB is perp. to EF.]

12. Let OA, OB, OC meet a str. line in A, B, C. The lines OA, OC, AC are in one plane (VI. 2.), and B lies in AC $\therefore$ the lines OA, OC, OB are in one plane.

13. Let MR be $\parallel$ to NO. Then MR is perp. to PQ (VI. 8.). But MO is perp. to PQ $\therefore$ PQ is perp. to the plane RMO (VI. 4.).

But the points R, M, N, O are in one plane since RM is ∥ to ON ∴ PQ is perp. to MN.

14. Let A, B be the given pts. Draw BD perp. to the given plane and produce it to C making DC = BD. Join AC meeting the plane at P. In the plane take any other point Q. Join PB, AQ, PD, QD, QC, QB. PB = PC and QB = QC (I. 4.). AQ + QB = AQ + QC > APC (I. 12.) *i.e.* > AP + PB ∴ the position found by joining AC is the one required.

15. Let PM, PN, PR perp. to OA, OB, OC be all equal; the angles POM, PON, POR are equal (I. 17), *i.e.* the locus of P is a str. line through O equally inclined to the 3 str. lines.

16. Let AB, CD be two str. lines not in a plane. If AC, BD were parallel, the pts. A, C, D, B would be in a plane ∴ AB, CD would be in a plane.

17. Proved in VII. 22.

18. In the figure of VI. 15., let the planes PQ, TX be ∥. $\frac{AE}{EB}=\frac{AH}{HD}$ (hyp.) ∴ EH is ∥ to BD and therefore ∥ to the plane TX (Ex. lxxii. 2.). Similarly HF is ∥ to the plane PQ and therefore to TX ∴ the plane RS is ∥ to the plane TX (VI. 13.).

19. By VI. 14. AB is ∥ to *ab* ∴ AB*ba* is a parm., and so is BC*cb*, etc. ∴ AB, BC, etc., are respectively equal to *ab*, *bc*, etc. ... Also ∠ABC = ∠*abc* (VI. 10.); and similarly for the other ∠s.

20. Produce CO to meet AB at F. The plane CPF which contains PO is perp. to the plane ABC (VI. 16.). Also it contains PC and is therefore perp. to the plane ABP ∴ it is perp. to the line AB (VI. 17.) ∴ COF is perp. to AB. Similarly AO is perp. to BC ∴ O is the orthocentre of △ABC.

21. In question 20 it is proved that the plane PCF is perp. to AB ∴ PF is perp. to AB ∴ the feet of the perps. from P coincide with the feet of the perps. from the vertices to the sides of △ABC. From the cyclic quadl. DOEC, ∠EDO = ∠ECO = 90° − ∠CAB. Similarly ∠FDO = 90° − CAB ∴ DA is the internal bisector of the ∠EDF ∴ BC, which is at rt. ∠s to it, is the external bisector.

22. Let BAC, EDF, HGK be the 3 ∠s. Make AB, AC, DE, DF, GH, GK all equal. Construct a △LMN with its sides MN, NL, LM equal to BC, EF, HK. Take O the circumcentre of LMN, and

draw OP perp. to the plane LMN. With centre L and radius equal to AB cut OP at P, and join LP, MP, NP. From $\triangle$s LOP, MOP, NOP by I. 4. LP = MP = NP $\therefore$ by I. 7. we can prove that the $\angle$s at P are equal to the given $\angle$s.

23. Let BAC, CAD, DAB form a trihedral $\angle$, AP being any line drawn within the trihedral $\angle$ to meet the plane BCD at P. Produce CP to meet BD at E. $\angle$CAD + $\angle$DAB = $\angle$CAD + $\angle$DAE + $\angle$EAB > $\angle$CAE + $\angle$EAB (VI. 18.). Similarly $\angle$CAE + $\angle$EAB > $\angle$CAP + $\angle$BAP $\therefore$ $\angle$CAD + $\angle$DAB > $\angle$CAP + $\angle$BAP. Similarly $\angle$DAB + $\angle$BAC > $\angle$DAP + $\angle$CAP and $\angle$BAC + $\angle$CAD > $\angle$BAP + $\angle$DAP $\therefore$ by adding and dividing by 2, $\angle$BAC + $\angle$CAD + $\angle$DAB > $\angle$BAP + $\angle$CAP + $\angle$DAP. The rest has been proved in question 2.

24. Let the planes intersect in AO. Let AB be a str. line perp. to AO, AC its projection; AD a line in the first plane not perp. to AO, AE its projection, BD being $\parallel$ to AO. In the right-angled $\triangle$s ABD, ACE, AB < AD, AC < AE, and BC = DE (II. 2.) $\therefore$ by superposing the right-angled $\triangle$s ABC, ADE we can prove $\angle$BAC gr. than $\angle$DAE.

25. Let AD be the common section of the given planes, AB, AC str. lines drawn in the planes perp. to AD. Draw BC perp. to AC. At B in the plane BAC make $\angle$CBE equal to the complement of the given $\angle$, BE meeting CA at E. Draw a circle with centre C and radius CE cutting AD at D. The right-angled $\triangle$s BCE, BCD are equal in all respects (I. 4.) $\therefore$ $\angle$CDB = $\angle$CEB = the given $\angle$ $\therefore$ CDB is the required plane. Impossible if the given $\angle$ is gr. than the $\angle$ between the planes.

26. Let O be the given pt.. AB, CD the given str. lines. Let the plane OAB cut CD at E. OE is the required line.

27. $OA^2 + OB^2 = AB^2$ (II. 11) $= AC^2$ (hyp.) $= OA^2 + OC^2$ (II. 11.) $\therefore$ OB = OC. Similarly OB = OA.

EXERCISES LXXIV.

2. Let P, Q, R be the centres of three adjacent faces, N the mid. pt. of the common edge of the first two faces, a the length of any edge. $PQ^2 = \frac{a^2}{4} + \frac{a^2}{4} = \frac{a^2}{2}$. Similarly $QR^2 = \frac{a^2}{2}$, and $RP^2 = \frac{a^2}{2}$ $\therefore$ PQR is an equilateral $\triangle$ whose side is $\frac{1}{2}a\sqrt{2}$ $\therefore$ the

new faces formed are equal equilateral $\triangle$s, and there are 8 of them $\therefore$ the figure is a regular octahedron.

3. Let ABCD, CDEF be adjacent faces. $AF^2 = AB^2 + BF^2 = AB^2 + BC^2 + CF^2$ (II. 11.) $= 3AB^2$.

4. Let ABCD, CDEF, EFGH be consecutive faces of a parallelepiped. Thus ADEH, BCFG are opposite faces. HE is equal and $\parallel$ to BC $\therefore$ HBCE is a parm. (II. 1.) $\therefore$ BE, HC bisect each other. HDCG is a parm. $\therefore$ DG, HC bisect each other. Thus we may prove that all the diagonals pass through the mid. pt. of HC and are bisected there.

5. Let ABCD, BCEF, EFGH be consecutive faces, so that AD, BC, FE, GH are parallel. Bisect at K, L, M, N, P, Q the sides BF, BC, CD, DH, HG, GF, which do not meet the diagonal AE. AM = AQ = EQ = EM (I. 4.) $\therefore$ AMEQ is a rhombus $\therefore$ MQ, AE bisect each other at rt. $\angle$s at O. Similarly KN, AE bisect each other at rt. $\angle$s at O, and LP, AE bisect each other at rt. $\angle$s at O $\therefore$ MQ, KN, LP are in one plane perp. to AE (VI. 5.). $KQ = \frac{1}{2}BG = \frac{1}{2}FH = QP$ (Ex. xx. 1.). Also $KQ = \frac{1}{2}BG$ and is $\parallel$ to BG $\therefore$ $KQ = \frac{1}{2}CH$ and is $\parallel$ to CH. But $MN = \frac{1}{2}CH$ and is $\parallel$ to CH $\therefore$ KQ is equal and $\parallel$ to MN. Similarly all the opposite sides of KLMNPQ are equal and parallel; and all the sides are equal, for each side = half the diagonal of a face $\therefore$ the figure is a regular hexagon.

6. Let A be the vertex, AH perp. to the plane BCDE, a the length of an edge. $AH = \sqrt{BA^2 - BH^2} = \sqrt{a^2 - \frac{a^2}{2}} = \frac{a\sqrt{2}}{2}$. When $a = 10$ cms., $AH = 5\sqrt{2} = 5 \times 1{\cdot}4142 = 7$ (to nearest cm.).

7. Let P, Q, R be points on edges which meet in A. $RP^2 + PQ^2 = AP^2 + AR^2 + AQ^2 + AP^2 = RQ^2 + 2AP^2 > RQ^2$ $\therefore$ RPQ is an acute angle. Similarly the other $\angle$s of the $\triangle$PQR are acute.

8. Let A, B be the fixed points, P one position of the moving point. Let C, D divide AB internally and externally in the given ratio. Then C, D are fixed points; also CP, PD are the bisectors of internal and external $\angle$s between AP, PB (V. 3. 4.) $\therefore$ $\angle$CPD is a rt. $\angle$ $\therefore$ P is at a constant distance from the mid. pt. of CD $\therefore$ the locus is a sphere with centre at the mid. pt. of CD.

9. Draw OM perp. to the plane. In OM take a point N such ON . OM = the given constant = OQ . OP $\therefore$ PQNM is a cyclic quadl. $\therefore$ $\angle$OQN = $\angle$PMN = 90° $\therefore$ Q is at a constant distance from the mid. pt. of ON $\therefore$ the locus of Q is a sphere, i.e. *the inverse of a plane is a sphere* (see page 369).

10. Draw AO from vertex A perp. to the face BCD. Let AEO be a plane perp. to BC meeting it at E. In the right-angled $\triangle$AOE, AE > OE $\therefore$ $\frac{1}{2}$AE . BC > $\frac{1}{2}$OE . BC, *i.e.* $\triangle$ABC > $\triangle$OBC. Similarly $\triangle$ACD > $\triangle$OCD, and $\triangle$ADB > $\triangle$ODB $\therefore$ the three faces which meet at A are together gr. than $\triangle$BCD.

11. Bisect BC at E. Join AE, DE. $\triangle$ABC is isosceles $\therefore$ AE is perp. to BC. CD = AB (hyp.) = AC (hyp.) = BD $\therefore$ $\triangle$DBC is isosceles $\therefore$ DE is perp. to BC $\therefore$ BC is perp. to the plane ADE (VI. 4.) and therefore to AD.

12. Let ABCD be a tetrahedron. Let a plane through AB perp. to CD meet CD at E. Then BE, AE are perp. to DC and so contain the orthocentres P, Q of $\triangle$s BDC, ADC $\therefore$ ABP is the plane required.

13. Proved in VII. 1.

14. If B be joined to P, Q, R the vertices of the base, the tetrahedron is divided into 3 equal parts. BC is the altitude and ARQ the base of one of these parts. But the base ARQ = the base PQR of the whole tetrahedron $\therefore$ altitude BC = $\frac{1}{3}$ of altitude AB. Also PB $= \frac{a\sqrt{3}}{3}$ where a = length of edge (VII. 5.) and BD . AP = 2$\triangle$ABP = AB . PB $\therefore$ $\frac{BD}{AB} = \frac{PB}{AP} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$ $\therefore$ $BD^2 = \frac{1}{3}AB^2$ $\therefore$ $BC^2 : BD^2 : BA^2 = \frac{1}{9}BA^2 : \frac{1}{3}BA^2 : BA^2 = 1 : 3 : 9$.

15. Let AB, AC, AD, etc., be the edges, AB being the greatest, and let AO be perp. to the base. $BO^2 = BA^2 - AO^2 > CA^2 - AO^2$ $\therefore$ BO > CO $\therefore$ by turning the $\triangle$AOC round AO into the plane AOB we can see that the $\angle$ABO < $\angle$ACO (I. 8.) Similarly $\angle$ABO < $\angle$ADO, and so on.

16. In the tetrahedron ABCD let E, F be the mid. pts. of BC, AD. Let a be the length of an edge. AE, DE are equal and are perp. to BC $\therefore$ their plane is perp. to BC $\therefore$ EF is

perp. to BC. But EF is perp. to AD since EAD is an isosceles $\triangle$. Also $EF^2 = AE^2 - AF^2 = AB^2 - BE^2 - AF^2 = a^2 - \frac{a^2}{4} - \frac{a^2}{4} = \frac{a^2}{2}$ $\therefore$ $EF = \frac{1}{2}a\sqrt{2} = \frac{1}{2}$ diagonal of the sq. on a.

17. Let BD, CD, CA, BA, edges of a tetrahedron, be bisected at E, F, G, H. Let P, Q, R, S be points in which the same edges are cut by a plane parallel to AD and BC. HE is ‖ to AD and is half of AD (Ex. xx. 1.). Similarly for GF. $\therefore$ HE is equal and ‖ to GF. Similarly EF, HG are ‖ to BC and each $= \frac{1}{2}$BC $\therefore$ HEFG is a parm. SR is in a plane with BC and does not meet BC (hyp.) $\therefore$ SR is ‖ to BC $\therefore$ $\frac{SR}{BC} = \frac{AS}{AB}$ (V. 5.) $= \frac{DP}{DB}$ (V. 2.) $= \frac{PQ}{BC}$ $\therefore$ SR = PQ. Similarly SP = RQ $\therefore$ SRQP is a parm. Again $\frac{SR}{BC} = \frac{AS}{AB}$ and $\frac{SP}{AD} = \frac{SB}{AB}$ $\therefore$ $\frac{SR \,.\, SP}{BC \,.\, AD} = \frac{AS \,.\, SB}{AB^2}$. Similarly $\frac{HG \,.\, HE}{BC \,.\, AD} = \frac{AH \,.\, HB}{AB^2}$ $\therefore$ $\frac{SR \,.\, SP}{HG \,.\, HE} = \frac{AS \,.\, SB}{AH \,.\, HB} = \frac{AH^2 - SH^2}{AH^2}$ $\therefore$ $\frac{\text{parm. SQ}}{\text{parm. HF}}$ = a ratio less than 1, *i.e.* the parm. HF is the maximum.

18. Let ABCD be a regular tetrahedron. Let PQRS be a plane section and let it be a parallelogram. Let P, Q, R, S lie on BD, CD, CA, BA. Since the plane SQ cuts the planes ABD, ACD in ‖ lines, it is ‖ to their common section AD. Similarly it is ‖ to BC $\therefore$ $\triangle$ASR is equilateral $\therefore$ SR = AS. Similarly SP = SB $\therefore$ the perimeter of parm. SQ = 2AB.

19. Let ABCD be a tetrahedron, and let a sphere touch its edges BA, BC, BD, CD, CA, AD in pts. E, F, G, H, K, L. AD + BC = AL + DL + BF + FC = AE + DH + BE + HC (tangents to a sphere) = AB + DC = (similarly) AC + BD.

20. Let E, F, G, H, K, L be the mid. pts. of AD, BC, CD, AB, AC, BD. HF is equal $\frac{1}{2}$AC and is ‖ to AC. EG is equal $\frac{1}{2}$AC and is ‖ to AC $\therefore$ HF is equal and ‖ to EG $\therefore$ HFGE is a parm. $\therefore$ HG passes through the mid. pt. of EF and is bisected there (II. 2. Cor. 3.). Similarly LK passes through the mid. pt. of EF and is bisected there.

21. See Question 5.

22. Let ABCD be such a tetrahedron. Draw DE perp. to BC meeting BC in E. Join AE. Draw BF perp. to DC meeting DC in F and DE in O. BC is perp. to ED and DA $\therefore$ BC is perp.

to the plane ADE (VI. 4. and 8.) $\therefore$ the plane ADE is perp. to the plane DBC. Similarly the plane ABF is perp. to the plane DBC $\therefore$ the intersection AO is perp. to the plane DBC (VI. 17.). But O is the orthocentre of DBC. Similarly the perp. from each vertex to the opposite face is the line joining the vertex to the orthocentre of that face. Draw BP, DR perp. to AF, AE. BP must intersect AO; for they lie in the plane AFB. DR must intersect AO; for they lie in the plane AED $\therefore$ the three perps. AO, BP, DR intersect each other. But they are not in one plane $\therefore$ they must be concurrent. Similarly the fourth perp. CQ meets BP on AO, and meets DR on AO $\therefore$ the 4 perps. are concurrent.

23. Let ABCD be any tetrahedron, E, F, G, H, K, L the mid. pts. of BC, AD, AB, CD, AC, BD. $4\mathrm{EF}^2 + \mathrm{AD}^2 = 2\mathrm{AE}^2 + 2\mathrm{DE}^2$ (IV. 12.) $= \mathrm{AB}^2 + \mathrm{AC}^2 - 2\mathrm{BE}^2 + \mathrm{DB}^2 + \mathrm{DC}^2 - 2\mathrm{BE}^2$ (IV. 12.) $= \mathrm{AB}^2 + \mathrm{AC}^2 + \mathrm{DB}^2 + \mathrm{DC}^2 - \mathrm{BC}^2$ $\therefore$ $4\mathrm{EF}^2 + \mathrm{AD}^2 + \mathrm{BC}^2 = \mathrm{AB}^2 + \mathrm{AC}^2 + \mathrm{DB}^2 + \mathrm{CD}^2$. Similarly $4\mathrm{GH}^2 + \mathrm{AB}^2 + \mathrm{CD}^2 = \mathrm{AD}^2 + \mathrm{BC}^2 + \mathrm{AC}^2 + \mathrm{BD}^2$, and $4\mathrm{KL}^2 + \mathrm{AC}^2 + \mathrm{BD}^2 = \mathrm{AB}^2 + \mathrm{BC}^2 + \mathrm{AD}^2 + \mathrm{CD}^2$ $\therefore$ by addition $4(\mathrm{EF}^2 + \mathrm{GH}^2 + \mathrm{KL}^2) =$ the sum of the sqs. on the six edges.

24. If a be the length of an edge, $\mathrm{A} = \dfrac{a^2\sqrt{3}}{4}$ $\therefore$ $a^2 = \dfrac{4\mathrm{A}\sqrt{3}}{3}$. But the altitude $= \dfrac{a\sqrt{6}}{3}$ (VII. 5.) $\therefore$ Volume $= \dfrac{1}{3} \cdot \dfrac{a\sqrt{6}}{3} \cdot \mathrm{A} = \dfrac{\mathrm{A}\sqrt{6}}{9} \cdot \dfrac{2\mathrm{A}^{\frac{1}{2}} \cdot 3^{\frac{1}{4}}}{\sqrt{3}} = \dfrac{2^{\frac{3}{2}} \cdot 3^{\frac{1}{4}}\mathrm{A}^{\frac{3}{2}}}{9}$.

25. In the tetrahedron ABCD, let AB be at rt. $\angle$s to CD, and AC at rt. $\angle$s to BC. Let E, F, G, H, K, L be the mid. pts. of BA, BD, BC, CD, CA, AD. EGHL is a parm. with its sides $\parallel$ to AC, BD $\therefore$ it is a rectangle $\therefore$ diagonal EH $=$ GL. Similarly FGKL is a rectangle $\therefore$ diagonal GL $=$ FK $\therefore$ in the parm. EFHK the diagonals EH, FK are equal $\therefore$ it is a rectangle. But its sides are $\parallel$ to AD, BC $\therefore$ AD, BC are at rt. $\angle$s. Also $\mathrm{AD}^2 + \mathrm{BC}^2 = 4\mathrm{FE}^2 + 4\mathrm{EK}^2 = 4\mathrm{FK}^2 = 4\mathrm{FG}^2 + 4\mathrm{GK}^2 = \mathrm{DC}^2 + \mathrm{BA}^2$. Similarly for $\mathrm{AC}^2 + \mathrm{BD}^2$.

26. Each of these planes contains one of the three joins of the mid. pts. of opposite edges. But there is one point common to all these joins (Question 20) $\therefore$ this point is common to all the six planes.

27. Take a plane through PQ a diagonal of a regular octahedron whose edge is a, and let it cut an edge at rt. $\angle$s in R. Draw QS perp. to PR produced. PRQ is the dihedral $\angle$ of the octahedron. $PR = RQ = \frac{a\sqrt{3}}{2}$; RT (the perp. from R to PQ) $= \frac{a}{2}$ $\therefore$ $\frac{1}{2}PQ = \sqrt{\frac{3a^2}{4} - \frac{a^2}{4}} = \frac{a}{\sqrt{2}}$ $\therefore$ $PQ = a\sqrt{2}$. By similar $\triangle$s $\frac{SQ}{PQ} = \frac{RT}{PR} = \frac{1}{\sqrt{3}}$ $\therefore$ $SQ = \frac{a\sqrt{2}}{\sqrt{3}} = \frac{a\sqrt{6}}{3}$ $\therefore$ $\triangle QRS$ is equal in all respects to $\triangle AEH$ in VII. 5. $\therefore$ $\angle AEH = \angle QRS =$ supplement of $\angle PRQ$, *i.e.* the dihedral $\angle$s of the regular tetrahedron and octahedron are supplementary.

EXERCISES LXXV.

1. Length reqd. $= \sqrt[3]{2000} = x$. $\log x = \frac{1}{3}(3{\cdot}30103) = 1{\cdot}1003$, $x = 12{\cdot}60$ ft. $= 12$ ft. $7{\cdot}2$ in.

2. Area of end $= \frac{1}{2} \times \frac{\sqrt{3}}{2}$ sq. ft. $\therefore$ vol. $= \frac{6}{2} \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} = 3({\cdot}866025) = 2{\cdot}598$ c. ft.

3. Let $a - b$, a, $a + b$ inches be the lengths of the sides. $3a = 15$ $\therefore$ $a = 5$, $(a-b)^3 + a^3 + (a+b)^3 = 495$ $\therefore$ $375 + 6 \times 5b^2 = 495$, $b^2 = 4$, $b = 2$. The sides are 3, 5, 7 inches respectively. The vols. are 27, 125, 343 cub. in.

4. Area reqd. $= \frac{\text{vol.}}{\text{length}} = \frac{1}{3000}$ sq. ft. $= \frac{144}{3000}$ sq. in. $= {\cdot}048$ sq. in.

5. Wt. of 1st. bar $= 96 \times 18$ lbs. Let $l =$ length, $d =$ thickness reqd. in inches. $2 \times 9 \times d =$ vol. of 2 in. of 2nd bar $= 27$ $\therefore$ $d = 1\frac{1}{2}$ in. Also $l \times 9 \times \frac{3}{2} =$ vol. of whole bar $= 96 \times 18$ $\therefore$ $l = \frac{96 \times 18 \times 2}{3 \times 9} = 10$ ft. 8 in.

6. Let the water rise x ft. $x \times 12 \times \frac{7}{2} = 195$, $x = \frac{65}{14}$ ft. $= \frac{32{\cdot}5}{7} = 4{\cdot}6428$ ft. $= 4$ ft. 8 in.

7. $BD = \sqrt{7^2 + 24^2} = 25$ in. $\therefore$ $CD = \sqrt{25^2 - 20^2} = 15$ in. Vol. of prism $= 18 \times ABCD = 18 \times \frac{1}{2}[7 \times 24 + 15 \times 20]$ c. in. $= 9 \times 12[14 + 25] = 12 \times 351 = 4212$ c. in. Area of ends $= 7 \times 24 + 20 \times 15$ sq. in. Area of faces $= 18(7 + 24 + 15 + 20)$ ... Total area $= 7 \times 24 + 20 \times 15 + 18 \times 66$ sq. in. $= 12[14 + 25 + 99] = 12 \times 138$ sq. in. $= 11\frac{1}{2}$ sq. ft.

8. Let $x =$ dist. between the $\|$ sides of the trapezium. $x^2 = 13^2 - 5^2 = 12^2 \therefore x = 12 \therefore$ area of base $= \frac{1}{2}(19 + 9)12 = 6 \times 28$ sq. in. Vol. $= \frac{7 \times \cancel{6} \times \overset{7}{\cancel{28}}}{\underset{6}{\cancel{1}\cancel{4}\cancel{4}}}$ c. ft $= 8\frac{1}{6}$ c. ft.

9. Vol. contained by side walls $= 85 \times 9 \times 14$ c. ft. Vol. cont. by roof $= \frac{1}{2} \times 7 \times 85 \times 9 \therefore$ total volume $= 85 \times 9 \times 7 \times \frac{5}{2}$ c. ft. $= 765 \times 7 \times \frac{10}{4} = \frac{53550}{4} = 13387\frac{1}{2}$ c. ft.

10. Area of hexagon $= 3 \times 4 \times \frac{\sqrt{3}}{2}$ sq. ft. Vol. of prism $= 6 \times 3 \times 2 \times \sqrt{3}$ c. ft. $= 6 \times 6(1{\cdot}73205)\ldots = 6 \times 10{\cdot}3923\ldots = 62{\cdot}354$ c. ft.

11. Vol. $= 9\pi \times 5 = \frac{9}{2}(31{\cdot}416)$ $[\pi r^2 h] = 9(15{\cdot}708) = 141{\cdot}372$ c. ft. $= 141$ to the nearest c. ft.

12. Suppose the last man uses x in. of the rad., the preceding man y in. and the first man z in, $\pi x^2 = \frac{1}{3}\pi(\frac{5}{2})^2$, $x = \frac{5}{2} \cdot \frac{\sqrt{3}}{3}$ ft. $= \frac{17{\cdot}3205}{12}$ ft. $= 17{\cdot}32$ in.; $\pi(x+y)^2 = \frac{2}{3}\pi(\frac{5}{2})^2$, $x + y = \frac{5}{2} \cdot \frac{\sqrt{6}}{3} = \frac{24{\cdot}49}{12}$ ft. $= 24{\cdot}49$ in.; $\therefore y = 7{\cdot}17$ in., $z = 30 - x - y = 5{\cdot}51$ in.

13. Let x inches $=$ a side of the cube. If ABCDEF is a right section of the tube, AC must be a diagonal of a face of the cube; $\therefore$ AC $= x\sqrt{2} = 2 \times 6 \sin 60° = 6\sqrt{3}$, $x = 3\sqrt{6}$ in. $= 3(2{\cdot}449) = 7{\cdot}35$ in. approx. A model greatly simplifies this question.

14. Let $h =$ ht., $r =$ rad. of base, $2\pi r = 10$, $\pi r^2 h = 600$, $h = \frac{600}{25} \cdot \pi = 24 \times 3{\cdot}1416 = 4 \times 18{\cdot}8496 = 75$ ft. to the nearest foot.

15. Vol $= \frac{1}{2}\frac{\sqrt{3}}{2} \cdot 6$ c. ft. $= \frac{3}{2}(1{\cdot}73205) = 3({\cdot}866025) = 2{\cdot}598$ c. ft. Number of gallons $= \frac{3\sqrt{3}}{2} \times \frac{1000}{16} \times \frac{1}{10} = \frac{2{\cdot}598075 \times 100}{16} = \frac{64{\cdot}9518}{4}\ldots = 16{\cdot}24$ gallons nearly.

16. Let r be the radius reqd. $\pi r^2 \times 11 = \pi 36 \times 12 + \pi 18^2 \times 6$, $r^2 = \frac{1}{11}(18 \times 12) \times (2 + 9) = 18 \times 12$, $r = 6\sqrt{6} = 6 \times 2{\cdot}449 = 14{\cdot}7$ in. nearly.

17. Let x be the ht. and $3x$ length of the slope of the part dug out. $9x^2 = x^2 + 70^2 \therefore 8x^2 = 70^2 \therefore x = \frac{70}{2\sqrt{2}} = \frac{70}{4}\sqrt{2}$. Vol. $= 70 \times \frac{1}{2} \times 70x = \frac{70^3\sqrt{2}}{8}$ c. ft. $= \frac{70^3}{8}(1{\cdot}41421) = \frac{70^2}{8}(98{\cdot}9947) = \frac{70}{8} \times 6929{\cdot}629 = \frac{485074{\cdot}03}{8} = 60634$ c. ft.

18. Let ABCD be a section of the cutting, AB being the base, BC the sloping side. Draw BE perp. to CD, CE $=$ EB $\tan 30° = \frac{8\sqrt{3}}{3}$. Vol. $=$ area of trapezium ABCD $\times 1000$ c. metres. Vol.

$= \frac{8}{2}(9{\cdot}4 + 9{\cdot}4 + \frac{8\sqrt{3}}{3})1000 = 8(9400 + \frac{4000\sqrt{3}}{3}) = 75200 + \frac{6928{\cdot}20}{3}$ $\times 8 = 75200 + 2309{\cdot}4 \times 8 = 75200 + 18475{\cdot}2 = 93675{\cdot}2$ c. metres.

EXERCISES LXXVI.

1. Area of end CD $= 5 \times 6$ sq. ft. Perp. from B on AC $= \frac{12\times\sqrt{3}}{2} = 6\sqrt{3}$. Vol. $= 5 \times 6 \times 6\sqrt{3} = 3 \times 6(17{\cdot}3205) = 3(103{\cdot}9230) = 311{\cdot}769$ c. ft.

2 Vol. $= \frac{25}{12} \times \frac{13}{2}$ c. ft. $= \frac{325}{24} = 13$ c. ft. 936 c. in.

3. Let AB be the length of pipe. Draw AC vertically and BC horizontally. Vol. = AC × area of horizl. section $= 4 \times \frac{13\times 12}{\sqrt{2}}$ c. in. $= 2 \times 13 \times 12 \times 1{\cdot}41421 = 2 \times 12 \times 18{\cdot}38473 = 2 \times 220{\cdot}61676$ $= 441{\cdot}233$ c. in.

4. Let r = rad. of circum-circle of the pentagon, a = a side of the pentagon, h = dist. of incentre from a side of the pentagon. $r^2 = \frac{2a^2}{5-\sqrt{5}}$ (p. 255) $= \frac{32(5+\sqrt{5})}{20} = \frac{8}{5}(5+\sqrt{5})$, $h^2 = r^2 - \frac{a^2}{4} = \frac{8}{5}(5+\sqrt{5})$ $- 4 = 4 + \frac{8\sqrt{5}}{5} = 7{\cdot}578$ $\therefore$ $h = 2{\cdot}752$. Area of ends $= \frac{2}{2} \,.\, 5 \,.\, ha =$ $20 \times 2{\cdot}752 = 55{\cdot}04$. Area of sides $= 5 \times 4 \times 7 = 140$ sq. ft. Total area $= 195{\cdot}04$ sq. ft.

5. Taking the length as 10 in. and the altitude as 8 in. Base × alt. = rt. sectn. × length $\therefore$ base $= \frac{340\times 10}{8} = 425$ sq. in. $= 2$ sq. ft. 137 sq. in.

EXERCISES LXXVII.

1. Let ABCD be the tetrahedron. Draw CHE perp. to BD and let AH be the alt. of the tetrahedron. $\text{CH} = \frac{2}{3}\text{CE} = \frac{2}{3} \cdot \frac{a\sqrt{3}}{2}$

$\therefore$ $\text{AH} = \sqrt{l^2 - \frac{a^2}{3}}$. Vol. $= \frac{1}{3}\text{AH} \times \frac{a^2}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{12}\sqrt{l^2 - \frac{a^2}{3}} \,.\, a^2$.

2. Alt. $= \sqrt{l^2 - \frac{a^2}{2}}$. Vol. $= \frac{a^2}{3}\sqrt{l^2 - \frac{a^2}{2}}$.

3. Let l = alt. of each slant face, h = alt. of pyramid. $l = \sqrt{h^2 + \frac{a^2}{4}}$ $\therefore$ $2\text{A} = al = a\sqrt{h^2 + \frac{a^2}{4}}$, $4\text{A}^2 = a^2\left(h^2 + \frac{a^2}{4}\right)$, $h^2 = \frac{4\text{A}^2}{a^2}$ $- \frac{a^2}{4}$, $h = \sqrt{\frac{4\text{A}^2}{a^2} - \frac{a^2}{4}}$. Vol. $= \frac{1}{3}a^2h = \frac{a^2}{3}\sqrt{\frac{4\text{A}^2}{a^2} - \frac{a^2}{4}}$.

4. Let h = alt. of pyramid. Rad. of circum-circle of base $= a \therefore h^2 = l^2 - a^2 \therefore h = \sqrt{l^2 - a^2}$. Vol. $= \frac{1}{3}\sqrt{l^2 - a^2} \cdot 3 \cdot a^2\frac{\sqrt{3}}{2} = \frac{a^2}{2}\sqrt{3(l^2 - a^2)}$.

5. Let x = rad. of circum-circle of base. $a^2 = 2x^2 - 2x^2\frac{1}{\sqrt{2}}$ (IV. 11.) $= x^2(2 - \sqrt{2})$, $x^2 = \frac{a^2}{2 - \sqrt{2}} = \frac{a^2(2 + \sqrt{2})}{2}$. Vol. $\frac{h}{3} \times 8 \times \frac{1}{2}x^2\frac{1}{\sqrt{2}} = \frac{8h}{6\sqrt{2}}\,\frac{a^2}{2}(2 + \sqrt{2}) = \frac{\sqrt{2}}{3}a^2h(2 + \sqrt{2})$, $l^2 = h^2 + x^2 = h^2 + \frac{a^2}{2}(2 + \sqrt{2})$.

6. As in the preceding, $x^2 = \frac{a^2}{2}(2 + \sqrt{2}) \therefore h^2 = l^2 - x^2 = l^2 - \frac{a^2}{2}(2 + \sqrt{2})$. Vol. $= \frac{\sqrt{2}}{3}a^2(2 + \sqrt{2})\sqrt{l^2 - \frac{a^2}{2}(2 + \sqrt{2})}$.

7. Let h = altitude of the pyramid. Area of base $= 3 \cdot a^2\frac{\sqrt{3}}{2}$ $\therefore$ $V = \frac{h}{3} \cdot 3a^2\frac{\sqrt{3}}{2}$, $h = \frac{2V}{a^2\sqrt{3}}$, $l = \sqrt{h^2 + a^2} = \sqrt{\frac{4V^2}{3a^4} + a^2}$.

8. If x = edge of base, $\frac{1}{3} \cdot h \cdot x^2 = V \therefore x^2 = \frac{3V}{h}$, $l = \sqrt{h^2 + \frac{x^2}{2}} = \sqrt{h^2 + \frac{3V}{2h}}$.

9. Let h = alt. of pyramid. $h^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}$, $h = \frac{a}{\sqrt{2}}$. Vol. $= \frac{1}{3}ha^2 = \frac{a^3}{3\sqrt{2}}$.

10. Let ABCD be the base, and O the vertex of the pyramid. Join DB, AC, cutting at G. OG $= h$. Let NMLK be a face of the cube, N lying in OA, M in OB, L in BG, K in AG. Also let x = edge of cube. $\frac{x}{h} = \frac{NK}{OG} = \frac{AK}{AG} = \frac{AG - KG}{AG} = 1 - \frac{KG}{AG} = 1 - \frac{x}{a}$. $x = \frac{ah}{a + h}$.

11. Area of PQR $= \frac{1}{4}\triangle$BDC (V. 11) $= \frac{1}{4} \cdot \frac{1}{2}\,\frac{a^2\sqrt{3}}{2} = \frac{a^2\sqrt{3}}{16}$. Perp. from O on PQR $= \frac{1}{2}h = \frac{\sqrt{6}a}{6}$ (VII. 5.) $\therefore$ vol. OPQR $= \frac{1}{3}\,\frac{a^2\sqrt{3}}{16} \times \frac{\sqrt{6}a}{6} = \frac{\sqrt{2}a^3}{96}$.

12. Let B, C, D be the mid. pts. of the edges which are conterminous at A. Vol. $= \frac{1}{3}$ AC $\times \triangle$ DAB $= \frac{a}{6} \times \frac{1}{2} \frac{a^2}{4} = \frac{a^3}{48}$.

13. If O is the centre, and POQ a diagonal of the base, draw from an angular pt. A of the top face AN perp. to POQ. OQ $= \frac{a}{\sqrt{2}}$, ON $= \frac{b}{\sqrt{2}}$. Let $h =$ alt. of frustum; $h^2 = c^2 - \left(\frac{a-b}{\sqrt{2}}\right)^2$ $= c^2 - \frac{(a-b)^2}{2}$; $\therefore$ vol. $= \frac{1}{3}\sqrt{c^2 - \frac{(a-b)^2}{2}}\sqrt{a^2+ab+b^2}$.

14. Thro. A and A′ ends of the top ridge draw vertl. planes APQ, A′P′Q′ perp. to the longer sides and cutting them in P, Q, and P′, Q′. Let APQ cut off pyramid APCBQ from the end of the bank. Draw AN perp. to PQ, and AM perp. to BC. MN bisects BC and PQ. Also $\angle$AMN $= \angle$APN (hyp.) $\therefore$ from $\triangle$s AMN, APN, MN $=$ PN $= b$ $\therefore$ vol. $= 2 \times$ pyramid APCBQ $+$ triangular prism between A and A′. Vol. $= \frac{2}{3}h \,.\, 2b^2 + h \,.\, b(2a - 2b) = \frac{2hb}{3}[2b + 3a - 3b] = \frac{2hb}{3}(3a - b)$.

15. $\frac{\text{The pyramid cut off}}{\text{The whole pyramid}} = \frac{1}{2}$ $\therefore$ $\frac{\text{alt. of pyramid cut off}}{\text{alt. of whole pyramid}} = \frac{1}{\sqrt[3]{2}}$ (VII. 14.) $\therefore$ the plane must divide the altitude (measured from the vertex) in the ratio of $1 : \sqrt[3]{2} - 1$.

16. With the same construction as in Example 14 above, $h = 10 \tan 40°$. Vol. $= \frac{2}{3} \,.\, 10 \tan 40° \times 10 \times 20 + \frac{1}{2}\, 10 \tan 40° \times 20 \times 80 = 4000 \tan 40°(\frac{1}{3} + 2) = 4000 \times \frac{7}{3} \times \cdot 8391 = \frac{4000}{3} \times 5\cdot 8737$ $= 4000 \times 1\cdot 9579 = 7831\cdot 6$ c. ft.

17. Let ABCD be the pyramid, BCD being an equilateral $\triangle$, and the $\angle$s at A rt. $\angle$s. Let p_1, p_2, p_3 be the perps. from any pt. in BCD on the other faces. $\triangle$s ACB, ACD, ABD are equal in area. And $p_1 \times \triangle$ ACB $+ p_2$ $\triangle$ ABD $+ p_3$ $\triangle$ ACD $= 3$ vol. of pyramid $\therefore$ $p_1 + p_2 + p_3 = \frac{3 \text{ vol.}}{\triangle \text{ACB}}$, which is constant.

18. Sum of perps. $\times$ area of any face $= 3$ vol. of whole figure.

19. Alt. $= \frac{\sqrt{6}a}{3}$ (VII. 3.) $\therefore$ vol. $= \frac{1}{3} \cdot \frac{\sqrt{6}a}{3} \cdot \frac{1}{2} \frac{a^2\sqrt{3}}{2} = \frac{a^3\sqrt{2}}{12}$.

20. If OA, OB, OC, OD are four conterminous edges, ABCD is a sq. Draw ON perp. to DB. Let $ON = h$, $DN = \frac{a\sqrt{2}}{2}$ $\therefore h^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}$, $h = \frac{a}{\sqrt{2}}$. Vol. $= \frac{2}{3}h \cdot a^2 = \frac{\sqrt{2}a^3}{3}$.

21. As in the preceding, diagonal $= 2h = \frac{2a}{\sqrt{2}} = \sqrt{2}a$.

22. Let OABCD be the pyramid on the sq. base ABCD. Draw OE perp. to BC, and let $OE = l$. Then $l^2 = 8^2 + 6^2 = 10^2$ $\therefore$ area of face $= \frac{1}{2} \times 12 \times 10 = 60$ sq. ft.

23. Let OABCD be the pyramid on a sq. base ABCD. Draw OE perp. to BC, and let $OE = l$, and $h =$ alt. of pyramid. Then $l^2 = 16^2 - 6^2 = 2^2(8^2 - 9) = 2^2(55)$, $h^2 = l^2 - 6^2 = 2^2(55 - 3^2) = 2^2 \times 46$, $h = 2\sqrt{46} = 2 \times 6{\cdot}782 = 13{\cdot}564$ ft. Vol. $= \frac{1}{3}h \cdot 144 = 48h = 12 \times 54{\cdot}256 = 651{\cdot}072$ c. ft.

24. If O is the vertex of the pyramid, AB one edge of the base, P the centre of the base and PN perp. to AB. $PN = 3\sqrt{3}$ in. from $\triangle$ APB $\therefore ON^2 = 9^2 + (3\sqrt{3})^2$ $\therefore ON = \sqrt{108} = 6\sqrt{3}$. Slant surface $= \frac{6}{2} \times 6 \times 6\sqrt{3} = 187{\cdot}06$ sq. in.

25. Let $h =$ alt. of pyramid. Taking a slant edge as 20 in. long, $h^2 = 20^2 - 12^2 = 4^2 . 4^2$, $h = 16$. Vol. $= \frac{6}{3}h \cdot \frac{12^2}{2}\frac{\sqrt{3}}{2} = 72\sqrt{3}h$ $= 1152\sqrt{3} = 1995{\cdot}26$ c. in.

26. Vol. of whole pyramid $= \frac{1}{3} \times 7^2 \times 8$ $\therefore$ if $V =$ vol. of pyramid cut off, $\frac{3V}{7^2 \times 8} = \frac{2^3}{8^3} = \frac{1}{64}$. $V = \frac{49}{24}$ c. in. Vol. of frustum $= \frac{7^2 \times 8}{3} - \frac{49}{24} = \frac{7(448 - 7)}{24} = \frac{7 \times 441}{24} = \frac{7 \times 147}{8} = \frac{1029}{8} = 128{\cdot}625$ c. in.

27. Let $2x$ be any diagonal. $2x^2 = 36$ $\therefore x = 3\sqrt{2}$. Vol. $= \frac{2}{3} . 3\sqrt{2} . 36 = 72\sqrt{2} = 72(1{\cdot}41421) = 8(12{\cdot}72789) = 101{\cdot}823$ c. in.

28. Vol. of pyramid $= \frac{1}{3} \times (\frac{9}{2})^2 \times \frac{15}{2}$ c. ft. Vol. of frustum $= \frac{61}{3}\left[(\frac{15}{2})^2 + \frac{15}{2} \cdot \frac{9}{2} + \frac{81}{4}\right] = \frac{61}{3 \times 4}(225 + 135 + 81) = \frac{61}{3 \times 4}(441) = \frac{61 \times 147}{4} = \frac{8967}{4}$ $\therefore$ total vol. $= 2241{\cdot}75 + \frac{81 \times 5}{8} = 2241{\cdot}75 + 50{\cdot}625 = 2292{\cdot}375$ c. ft. Weight $= \frac{2292{\cdot}375 \times 17\cancel{0}}{224\cancel{0}}$ tons $= 173{\cdot}975$ tons, nearly.

29. Surface $= \frac{4}{2} 4^2 \frac{\sqrt{3}}{2} = 16\sqrt{3} = 16(1{\cdot}73205) = 4(6{\cdot}92820) = 27{\cdot}713$ sq. in.

30. If l = a slant edge, $l^2=(\frac{15}{2})^2+(5\sqrt{2}-\frac{5\sqrt{2}}{2})^2=(\frac{5}{2})^2(9+2)$ $\therefore$ $l=\frac{5}{2}\sqrt{11}=8{\cdot}29$ in. Let x = alt. of a face, $l^2=x^2+(\frac{5}{2})^2$ $\therefore$ $x^2=\frac{25}{4}(11-1)=\frac{250}{4}$. Area of slant face $=\frac{1}{2}(10+5)\frac{5\sqrt{10}}{2}$ $=\frac{75\sqrt{10}}{4}=59{\cdot}29$ c. in.

31. (1) Let the plane CDEF cut off the angular prism ACDEFB. Let a = edge of cube, AC $=b$, AD $=c$, BF $=d$, BE $=e$. Vol. cut off = a frustum of a pyramid $=\frac{1}{3}a\left[\frac{1}{2}bc+\frac{1}{2}\sqrt{bcde}+\frac{de}{2}\right]$ $=\frac{a}{6}[bc+\sqrt{bcde}+de]$.

(2) Let ABCD be a face, EFGH the cutting plane, AE, DF, CG BH being parallel edges. Vol. = Pyramid FGCBH + pyr. FABCD + pyr. FABHE $=a\times$ area GCBH + ED area ABCD $+\,a\times$ area ABHE $=\frac{a}{3}\left[\frac{a(d+e)}{2}+c\,.\,a+\frac{a(b+d)}{2}\right]=\frac{a^2}{6}(b+2c+2d+e)=$ $\frac{a^2}{2}(b+e)=\frac{a^2}{2}(c+d)$.

EXERCISES LXXVIII.

1. Cone cut off $=\frac{1}{8}$ of whole cone (VII. 14.) $\therefore$ frustum $=\frac{7}{8}$ of whole cone $=\frac{7}{8}\cdot\frac{1}{3}\cdot\pi r^2h=\frac{7\pi r^2h}{24}$.

2. Alt. $=\frac{\sqrt{6}a}{3}$ (VII. 5.) $\therefore$ vol. of frustum $=\frac{7}{8}$ of tetrahedron (VII. 14.) $=\frac{7}{8}\frac{1}{3}\frac{\sqrt{6}a}{3}\cdot\frac{1}{2}\frac{a^2\sqrt{3}}{2}=\frac{7\sqrt{2}a^3}{96}$.

3. $\frac{\text{Vol. of cone cut off}}{\text{Vol. of whole cone}}=\frac{2^3}{5^3}$ (VII. 14.) $\therefore$ vol. of cone cut off $=\frac{2^3}{5^3}\cdot\frac{1}{3}\cdot\pi r^2h=\frac{8\pi r^2h}{375}$.

4. $\frac{\text{Surface of cone cut off}}{\text{Surface of whole cone}}=\frac{2^2}{5^2}=\frac{4}{25}$ $\therefore$ curved surface of frustum cut off $=\frac{21}{25}$ curved surface of whole cone (V. 12.) $=\frac{21}{25}\pi rl=\frac{21}{25}\pi r\sqrt{h^2+r^2}$.

5. Vol. generated is two cones on a common base of rad. h. If x is the alt. of one cone, $a-x$ is the alt. of the other $\therefore$ vol. $=\frac{1}{3}\pi h^2(a-x)+\frac{1}{3}\pi h^2x=\frac{1}{3}\pi h^2a$.

6. Alt. of $\triangle = \frac{2A}{a}$ $\therefore$ as in the preceding example, vol. $= \frac{1}{3}\pi\left(\frac{2A}{a}\right)^2 . a = \frac{4}{3}\pi\frac{A^2}{a}$.

7. Let $h =$ the perp. from the rt. $\angle$ upon the hypotenuse. $h = \frac{bc}{a}$ $\therefore$ as in the preceding, vol. generated $= \frac{1}{3}\pi h^2 a = \frac{1}{3}\pi\frac{b^2c^2}{a}$.

8. The vol. generated is a frustum of a cone, radii of ends b and c; alt. a $\therefore$ vol. $= \frac{\pi a}{3}[b^2 + bc + c^2]$ (VII. 13.).

9. Alt. of cone $= \frac{r}{\sqrt{3}}$ $\therefore$ vol. $= \frac{1}{3}\pi r^2 h = \frac{\pi r^3}{3\sqrt{3}}$.

10. Alt. of cone $= \sqrt{l^2 - r^2}$ $\therefore$ vol. $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2}$.

11. Let $x =$ alt. of frustum, $l' =$ length of its slant side. $l' = \sqrt{241}$ by VII. 18. $241 = x^2 + 16$, $x = 15$ ft.

12. Radius of base $= 1$ in. $\therefore$ length of slant side $= \operatorname{cosec} 15^\circ$ in. $\therefore$ surface $= \pi r l = \frac{\pi}{\sin 15^\circ} = \pi\frac{2\sqrt{2}}{\sqrt{3} - 1} = \pi\sqrt{2}(\sqrt{3} + 1) = \pi(\sqrt{6} + \sqrt{2})$ $= 12{\cdot}14$ sq. in.

13. If $r =$ rad. of base and $l =$ slant side, $\pi r^2 = 180$ sq. ft. Area of canvas $= \pi r l = \pi r\sqrt{r^2 + (\frac{15}{2})^2} = \frac{22}{7} . \sqrt{\frac{180 \times 7}{22}(\frac{180 \times 7}{22} + \frac{15^2}{4})}$ $= \frac{15 . 22 . 6}{7} . \sqrt{\frac{7}{44} . \frac{(56 + 55)}{22}} = \frac{15 . 6}{7}\sqrt{388{\cdot}5} = \frac{90}{7} \times 19{\cdot}71 = 253{\cdot}4$ sq. ft.

14. If $r =$ rad. of base, $r = \frac{14}{\sqrt{3}} = \frac{14\sqrt{3}}{3}$ in. Vol. of whole cone $= \frac{1}{3}\pi(\frac{14^2}{3})14 = \frac{1}{3} \times \frac{22}{7} \times \frac{14^3}{3} = \frac{1}{3} \times \frac{176 \times 7^2}{3} = \frac{1}{3} \times \frac{1232 \times 7}{3} = \frac{1}{3} \times \frac{8624}{3}$ $= \frac{1}{3}\ 2874{\cdot}67$ c. in. Vol. of cone cut off $= \frac{1}{8}(\frac{2874{\cdot}67}{3}) = \frac{359{\cdot}33}{3}$ c. in. $= 119{\cdot}78$ c. in. Vol. of frustum $= \frac{2515{\cdot}34}{3}$ c. in. $= 838{\cdot}45$ c. in. Slant surface of whole cone $= \pi r l = \frac{22}{7} . \frac{14\sqrt{3}}{3}\sqrt{\frac{14^2}{3} + 14^2} =$ $\frac{22}{7} . \frac{14^2}{3} . 2 = 14 \times \frac{88}{3} = \frac{1232}{3}$ sq. in. Slant surface of cone cut off $= \frac{1}{12}(1232) = 102{\cdot}67$ sq. in. Slant surface of frustum cut off $= 308$ sq. in.

15. Outside curved surface $= 2\pi . \frac{5}{2} . 4 = 20\pi$ sq. in. Outside plane surface $= \pi\left(\frac{5}{2}\right)^2 = \frac{25\pi}{4}$ sq. in. Inside surface $=$

$\pi\frac{5}{2}\sqrt{16+\frac{25}{4}}=\pi\frac{5}{4}\sqrt{89}=\pi\frac{10}{8}\times 9{\cdot}434=\frac{11}{7}\times\frac{1}{4}\times 94{\cdot}34=\frac{1037{\cdot}74}{7\times 4}$ $=\dfrac{259{\cdot}44}{7}=37{\cdot}06$ sq. in. Whole surface $=\dfrac{105\pi}{4}+37{\cdot}06=$ $\frac{15\times 11}{2}+37{\cdot}06=82{\cdot}5+37{\cdot}06=119{\cdot}56$ sq. in.

16. Vol. $=\dfrac{\pi}{3}(5\sqrt{3})^2 10=\dfrac{750\pi}{3}=\dfrac{1500\times 11}{3\times 7}=\dfrac{16500}{3\times 7}=786$ c. in

17. Proved in VII. 18.

18. $r=$ rad. of whole cone, $r'=$ rad. of cone cut off; $h=$ ht. of whole cone, $h'=$ ht. of cone cut off; $l=$ length of slant side, $l'=$ length of slant side cut off; $\pi r'l'+\pi r'^2=\pi rl-\pi r'l'+\pi r'^2+\pi r^2$; $2r'l'=rl+r^2$; $\dfrac{l'}{l}=\dfrac{r'}{r}$ $\therefore$ $\dfrac{2l'^2r}{l}=rl+r^2$; $\dfrac{l'^2}{l^2}=\dfrac{r+l}{2l}$; $\dfrac{l'}{l}=\dfrac{\sqrt{r+l}}{\sqrt{2l}}$. Reqd. ratio $=\dfrac{l'}{l-l'}=\dfrac{\sqrt{r+l}}{\sqrt{2l}-\sqrt{r+l}}$.

EXERCISES LXXIX.

1. By symmetry rad. $=\dfrac{1}{4}$ alt. $=\dfrac{\sqrt{6}a}{12}$

2. Rad. $=\dfrac{3}{4}$ alt. $=\dfrac{3}{4}\,\dfrac{\sqrt{6}a}{3}=\dfrac{a\sqrt{6}}{4}$.

3. Rad. of section $=\sqrt{10^2-6^2}=8$ ft. Area $=\pi 64=64(3{\cdot}1416)=201{\cdot}0624$ sq. ft.

4. Area of cap. $=2\pi\times 10\times 2$ (VII. 23. Cor.). Area of circ. sectn. $=\pi 6^2$. Total area $=\pi(40+36)=76\pi=238{\cdot}76$ sq. ft.

5. Let $x=$ rad. of the common section. $10x=8\times 6$. Area of section $=\pi x^2=\dfrac{\pi\times(48)^2}{100}=72{\cdot}38$ sq. ft.

6. Let $r=$ rad. of sphere. $4\pi r^2=1000$ $\therefore$ area of section $=\pi(r^2-25)=250-25\pi=171{\cdot}46$ sq. ft.

7. Let $x=$ rad. of inner surface. $\frac{4}{3}\pi x^3\frac{66}{10}=\left[\frac{4}{3}\pi(6^3-x^3)\right]\frac{42}{10}$, $x^3(66+42)=6^3\times 42$, $x^3=\frac{6^2\times 42}{18}=6\times 14$. $\log x=\frac{1}{3}(1{\cdot}9243)=\cdot 6414$, $x=4{\cdot}379$ in. Thickness of iron $=1{\cdot}62$ in.

8. Wt. $=\frac{4}{3}\pi(11^3-9^3)\frac{7776}{1728}$ ozs. $=3{\cdot}1416(1331-729)6=3{\cdot}1416\times 602\times 6=709{\cdot}22$ lbs.

9. Let a = edge of cube, r = rad. of sphere. V_1 = vol. of cube, V_2 = vol. of sphere. $6a^2 = 4\pi r^2$, $\frac{a^2}{r^2} = \frac{4\pi}{6}$. $\frac{V_1}{V_2} = \frac{a^3}{\frac{4}{3}\pi r^3}$ $= \frac{3}{4\pi} \cdot \left(\frac{2\pi}{3}\right)^{\frac{3}{2}} = \frac{3}{4}\left(\frac{8\pi}{27}\right)^{\frac{1}{2}} = \frac{3}{4}(\cdot 9308)^{\frac{1}{2}} = \frac{3}{4}(\cdot 9648) = \cdot 72$ nearly, *i.e.* $\frac{V_1}{V_2} = \frac{72}{100}$.

10. Vol. of iron $= \frac{4}{3}\pi[5^3 - 4^3] = \frac{4}{3}\pi(125 - 64) = \frac{4 \times 61\pi}{3}$ c. in.
Wt. $= \frac{4 \times 61 \times \pi}{3} \times \frac{1000}{1728} \times \frac{721}{100}$ ozs. $= \frac{6710 \times 103}{3 \times 216}$ ozs. $= 66\cdot 66$ lbs.

11. (1) Let x in. be the thickness of gold. $\frac{4}{3}\pi[\overline{2+x}|^3 - 2^3]$ $= \frac{4}{3}\pi 2^3$, $\overline{2+x}|^3 = 2 \times 2^3$, $x = 2[\sqrt[3]{2} - 1] = 2(\cdot 26) = \cdot 52$ in. $[\log 2^{\frac{1}{3}}$ $= \frac{1}{3}(\cdot 3010) = \cdot 1003 = \log 1\cdot 260]$.
(2) $4\pi(2+x)^2 = 4\pi \,.\, 2^2 \times 2$, $2 + x = 2\sqrt{2}$, $x = 2(\sqrt{2} - 1) = 2(\cdot 414)$ $= \cdot 83$ in.

12. Let O be the centre of the earth, B the pt. at an alt. of 2000 ft., BA, BD tangents to the centre. Let OB meet AD at C. OC . OB = OA². Hence if h = alt. of the visible segment. $4000 - h$ $= \frac{4000^2}{4000 + \frac{2000}{5280}}$. $h = 4000 - \frac{4000^2}{4000 + \frac{2000}{5280}} = \frac{8{,}000{,}000}{4000 \times 5280 + 2000}$ $= \frac{4000}{10561}$. Fraction reqd. $= \frac{2\pi \,.\, 4000h}{4\pi 4000^2} = \frac{h}{8000} = \frac{1}{21122}$.

13. Let d = diamr. reqd. Vol. melted down $= 1^3 - \frac{4}{3}\pi(\frac{1}{2})^3$ $\therefore \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = 1 - \frac{4}{3} \times \frac{\pi}{8}$, $\frac{d^3}{8} = \frac{3}{4\pi} - \frac{1}{8}$, $d^3 = \frac{6}{\pi} - 1 = 6 \times \cdot 31831 - 1 =$ $\cdot 90986$. $\log d = \frac{1}{3}(\bar{1}\cdot 9590) = \bar{1}\cdot 9863 = \log \cdot 9690$, $d = \cdot 969$ ft. $=$ $11\cdot 63$ in.

14. Let $2r$ = the diameter reqd. $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 7^3 - \frac{4}{3}\pi 5^3$ $\therefore r^3$ $= 7^3 - 5^3 = 343 - 125 = 218$. $\log r = \frac{1}{3}(2\cdot 3385) = \cdot 7795 = \log$ $6\cdot 019$, $r = 6\cdot 019$ in., diam. $= 12$ in.

15. If r = rad. of the sphere, $r = 6 \tan 30° = 2\sqrt{3}$ in. Surface $= 4\pi r^2 = 48\pi = 48(3\cdot 1416) = 4(37\cdot 6992) = 150\cdot 80$ sq. in.

16. Let r = rad. of sphere. $4\pi r^2 = \pi(\frac{5}{2})^2$ $\therefore r = \frac{5}{4}$ ft. Vol. $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\frac{125}{64} = \frac{125}{3 \times 16} \,.\, \pi = \frac{3141\cdot 6}{3 \times 16 \times 8} = \frac{1047\cdot 2}{16 \times 8} = \frac{130\cdot 9}{16} = \frac{32\cdot 7}{4} =$ $8\cdot 2$ c. ft.

17. Let x = dist. reqd., h = ht. of visible segment, O the centre of the sphere, D the pt. of observation, DA and DE tangents to the sphere. Also let OD cut the sphere at C, and AE at B. $2\pi 5h = \frac{3}{8} 4 \,.\, \pi \,.\, 25 \;\therefore\; h = \frac{15}{4}$ ft., $OB = \frac{5}{4}$ ft. $OB \,.\, OD = OA^2 \;\therefore\; (\frac{5}{4})(5 + x) = 25,\; 5 + x = 20,\; x = 15$ ft.

18. Let r = rad. reqd. $\dfrac{\pi r^2}{100} = \dfrac{4}{3}\pi\left(\dfrac{1}{2}\right)^3,\; r^2 = \dfrac{100}{6} = 16{\cdot}67,\; r = 4{\cdot}08$ in.

19. Let O be the centre of the earth, AD a diamr. of the ice field, and let the bisector OC of the $\angle$AOD meet AD at B. $BC = 4000 - OB = 4000 - 4000 \cos 5° = 4000({\cdot}003805) \;\therefore$ area reqd. $= 2 \times \frac{22}{7} \times 4000^2 \times {\cdot}003805 = \frac{2678720}{7} = 382674\frac{2}{7}$ sq. miles.

20. Let a = an edge of the cube. $a^3 = \frac{4}{3}\pi[5^3 - 3^3] = \frac{4}{3}\pi \,.\, 98 = \frac{4}{3} \times 98 \times 3{\cdot}1416 = 410{\cdot}5|024$ c. ft. $\log a = \frac{1}{3}(2{\cdot}6133) = {\cdot}8711 = \log 7{\cdot}432,\; a = 7{\cdot}43$ ft.

21. Let r = rad. of rim of umbrella, O the centre of the sphere of which the umbrella is a segment, h the ht. of the segment. $2\pi \,.\, \frac{7}{2}h = \frac{44}{3} \;\therefore\; h = \frac{44}{3} \,.\, \frac{1}{22} = \frac{2}{3}$ ft. Draw ON perp. to the circle formed by the rim. $ON = \frac{7}{2} - \frac{2}{3} = \frac{17}{6}$ ft. Area reqd. $= \pi r^2 = \pi[(\frac{7}{2})^2 - (\frac{17}{6})^2] = \frac{22}{28}[49 - \frac{289}{9}] = 13{\cdot}27$ sq. ft.

22. Let O be the centre of the sphere, A, B, C angular pts. of one end of the prism, OG perp. to ABC, AGN perp. to BC. $AG = \frac{2}{3} AN = \frac{2}{3} \,.\, AC \sin 60° = \frac{\sqrt{3}}{3}$ ft. $OG^2 = AO^2 - AG^2 = 1 - \frac{1}{3} = \frac{2}{3}$ $OG = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$ ft. Ht. of prism $= \frac{2\sqrt{6}}{3}$ ft. Vol. of prism $\frac{1}{2} \,.\, \frac{\sqrt{3}}{2} \times \frac{2\sqrt{6}}{3} = \frac{\sqrt{2}}{2}$ c. ft. $= 1221{\cdot}88$ c. in.

23. Alt. of tetrahedron $= \frac{10\sqrt{6}}{3}$ cms. Vol. of tetra. $= \frac{1}{3}\,\frac{100}{2}\,\frac{\sqrt{3}}{2} \times \frac{10\sqrt{6}}{3}$ c. cms. $= \frac{1000\sqrt{2}}{12}$. Let n be the no. of bullets. $n \times \dfrac{4}{3}\pi\left(\dfrac{1}{2}\right)^3 = \dfrac{1000\sqrt{2}}{12},\; n = \dfrac{1000\sqrt{2}}{2\pi} = \dfrac{1414{\cdot}21}{2} \times {\cdot}31831 = 225.$

24. Let h = ht. of cone. $\frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^3 \;\therefore\; h = 2r.$

25. Mass $= \frac{4}{3}\pi[6^3 - 4^3]\frac{504}{1728}$ lbs. $= \pi 19 \times \frac{28}{9} = 3{\cdot}1416 \times \frac{19 \times 28}{9} = 186$ lbs.

ADDITIONAL EXERCISES XVII.

29. Draw AB 5 cms. Make $\angle$BAD 62°. Make AD 11 cms. From C the mid. point of BD draw CE perp. to BD to meet AD at E. The $\triangle$ABE is the one reqd. For its perimeter = DE + EA + AB = 16 cms. BE = 5·45 cms.; EA = 5·55 cms. approximately.

30. Draw AB 5 cms. Cut off AC equal to 4 cms. Draw CD 4 cms. perp. to AB. Join AD, BD. AD = 5·66 cms.; BD = 4·12 cms.

31. (*a*) The $\angle$ of a regular hexagon = 120° $\therefore$ 3 regular hexagons fit round a point without leaving any gap. Thus any number of equal regular hexagons can be fitted together to form a pavement.

(*b*) The $\angle$ of a regular octagon = 135° $\therefore$ two regular octagons and a square fit round a point. Fit together two equal regular octagons with a common side. One square with a side of the same length will fill the gap. Two other equal regular octagons may be applied to the remaining sides of the square, and so on.

32. The sum of the other two $\angle$s = 165°. Make any $\angle$BAC, say 70°. Make $\angle$ABC 95°. Then $\angle$C = 180° − 70° − 95° = 15°.

33. (*a*) Draw perps. to two sides from their mid. points. The intersection of these is the point equidistant from all the vertices (I. 23.). Distance 1·78 cms.

(*b*) Bisect two of the angles (or two exterior angles). The intersection of the bisectors is a point equidistant from all the sides (I. 24.). r = ·82 cms.

34. From D, the mid. point of AB draw DH perp. to AB Any point in DH is equidistant from A and B (I. 23.). With centre C and radius 2 cms. describe a circle cutting DH in E, F. These are the reqd. points.

35. Through E, the mid. point of BC draw a perp. DEF. Cut this perp at D, F by a circle with centre A and radius 2 inches. D, F are the required points. By measurement ED = EF = 1 inch. $\angle$BAD = 30°.

36. Heights in feet 10·35, 20, 28·28, 34·64. The increase of height is not proportional to the increase of angle of elevation.

37. $x=6$, $y=4$, $z=3$. Construct by I. 25.

38. Draw AB to represent 70 horizontal feet. Draw perps. DB, ACE. Let DB, AC each represent 20 feet. Make $\angle$CDE $56\frac{1}{2}°$. AE represents the flagstaff. Height 125·76 feet.

39. Draw AB vertical to represent 100 feet. Draw AD, BC horizontal. Make an angle DAC 20°. BC represents the breadth of river, 274·75 feet.

ADDITIONAL EXERCISES XXV.

39. The reqd. error = 3·4 feet.

40. (1) Ht. of tower = 93·3 yards.

(2) Find a pt. on the ground where the tree subtends 45°. The dist. of this pt. from the tree = the ht. of the tree.

41. 16·1 yards.

42. The diagonals bisect one another at rt. $\angle$s

43. Draw AB equal to 65 mms. $\angle$ABC, 70°. Draw AD || to BC, and with centre B and rad. 85 mms. describe an arc cutting AD at D. Draw DC || to AB. ABCD is the reqd. parm.

44. Take AB 12 cms. long and draw CD || to it, at a dist. of 3 cms. from it. With centres A and B and rad. 6 cms. describe arcs cutting CD at C and D. ACDB is the trapezium.

45. Make $\triangle$AOB such that AB = 33 mms., AO = 30 mms. and BO = 38 mms. Produce AO, BO to C and D, making OC equal to OA, and OD equal to OB. ABCD is the parm.

46. Draw lines || to those containing the $\angle$65°, and at distances of 1 in. and 2 in. from them. Their pt. of intersection is the reqd. pt.

47. Reqd. distance = 28·85 ft.

48. Let ABCDEF be the hexagon. Draw GAH || to CE to meet EF at H and CB at G. Rect. GHEC = hexagon in area.

From EH produced cut off EK = EC. Bisect EK at O, describe circle KPE, centre O, cutting HG at P. $EP^2 = EH . EK$ (proved in last line but four of II. 11.) = EC . EH = the hexagon. Area = 2·6 sq. in.

49. If ABCD is the sq., bisect AD at E, BC at F. Produce EF to H, making FH = FE. AHD is the △ reqd.

50. (1) Seven rectangles each 12 sq. cms. in area.
(2) Twelve. (3) 84.

51. By measurement, the longer diagonal = 7·39 in. Diagonal of sq. = 5·66. Reqd. shortening = 1·73 in. Area of rhombus = $8\sqrt{2} = 11·31$. Increase of area = 4·69 sq. in.

52. Correct area = 288 sq. ft. ∴ correct side = 16·97 ft. He must shorten each side by ·03 feet.

53. Let the pole BC break at O, and just miss the window D in the house AD. OC = OD. Let OC = x and EO be ∥ to AB. From △DEO, $(x-20)^2 + 40^2 = x^2$ (II. 11.) whence $x = 50$ feet. This problem might also be solved on sqd. paper by making ∠CDO = ∠DCO.

54. Let ABCDEF be the hexagon, and draw HFG ∥ to EA to meet DE at H and BA at G. As in problem 48 above, rect. HGBD = the hexagon. Produce BG to K, making GK = GB. KHB is the reqd. △. BK = 6 in.

55. If ABCD is the trapezium, AB being the largest side, draw DE perp. to AB. AE = 1 cm. ∴ DE = $\sqrt{AD^2 - AE^2} = \sqrt{8}$ = 2·83 cms.

56. If x is one side of the rect. $6 - x$ is the other side ∴ $x(6-x) = \frac{27}{4}$ whence $x = 4\frac{1}{2}$ or $1\frac{1}{2}$ approx. ∴ $4\frac{1}{2}$, $4\frac{1}{2}$, $1\frac{1}{2}$, $1\frac{1}{2}$ in. are the parts.

57. The △ formed will be half an equilateral △. Ht. = 10 feet. Horizontal dist. = 17·32 ft.

58. If x and $x + 24$ be the sides, $x(x+24) = 640$ whence $x = 16$ ∴ the sides are 16 and 40 cms.

59. Describe ABC an equilat. △ on a base 8. Bisect BC at D, AB at E. AD = $4\sqrt{3}$, AE = 4, ∠EAD = 30° ∴ we have to find a △, rt ∠d. and isos. and equal in area to EAD. Take F the mid. pt. of BD. △AFD = △AED. From DA cut off DH = DF.

On AD describe a semicircle, and let HK perp. to AHD meet it at K. $DK^2 = DH . DA$ as in problem 48 above $= 2\triangle ADF = 2\triangle AED$. But if x is the reqd. hypotenuse, $\triangle AED = \frac{x^2}{4}$ $\therefore$ $\frac{x^2}{4} = DK^2$ $\therefore$ $x = 2DK$.

60. Area = a rect. $3\frac{1}{2}$ ft. by $2\pi \times \frac{500}{4} = 2750$ sq. ft. taking $\pi = \frac{22}{7}$.

61. If x and $2x$ are the sides, $x^2 = 98$, $x = 7\sqrt{2}$. Hypotenuse $= x\sqrt{5} = 7\sqrt{10} = 22{\cdot}13$ cms. If p is the perp. reqd. $\frac{p}{2} \times 7\sqrt{10} = 98$, $p = \frac{28}{\sqrt{10}} = \frac{28\sqrt{10}}{10} = 8{\cdot}85$ cms.

62. Draw a str. line ABC, making $AB = 2{\cdot}5$ cms. AC $3{\cdot}6$ cms. On AC describe a semicircle, and let BD, perp. to ABC, meet it at D. AD is a side of the reqd. sq. For $\triangle$ADC is rt. $\angle$d. (Ex. xviii. 9.) and $AD^2 = AB . AC$ (II. 11. last line but four). The diagonal $= 4{\cdot}2$ cms.

63. Alt. $= 2\sqrt{3}$ (II. 11.) $= 3{\cdot}46$ cms..

64. Length of each side $= 1{\cdot}008$ of the given line. Diff. $= {\cdot}008$ of the given line. Fraction $= \frac{1}{125}$ about.

65. Area $= \frac{1}{2} \times 84 \times (37 + 41) = 42 \times 78 = 3276$ sq. ft.

66. Let ABCD be the fig. $AB = 28$, $BC = 25$, $CD = 3$, $AD = 30$ cms. Draw DE, CF perp. to AB. Let $DE (= CF) = x$, $28 = AE + EF + FB = \sqrt{30^2 - x^2} + 3 + \sqrt{25^2 - x^2}$. Whence $x = 24$ cms. Area $= \frac{1}{2}(28 + 3)24 = 31 \times 12 = 372$ sq. cms.

67. The two $\triangle$s formed are equal in all respects. Width of road $= 18 + 24 = 42$ feet. Length of ladder $= \sqrt{18^2 + 24^2} = 6\sqrt{3^2 + 4^2} = 30$ ft.

68. Length of median $= 13$.

69. Area $= 6$ times an equilat. $\triangle$ of sides $16 = 665{\cdot}1$ sq. units.

70. (*a*) Area $= 13{,}403$ sq. yds. }
(*b*) Area $= 136{,}350$ sq. yds. } See p. 98.

71. The angles are equal to those of $\triangle$ABC. If we turn ABC thro. a rt. $\angle$, its sides become $\parallel$ to those of the new $\triangle$.

72. $PM + PN = 4$ in. $= OA = OB$, for $\triangle$s AMP, BNP are isos. and rt. $\angle$d.

73. Draw AE perp. to BD. AE bisects BD. BD $= 16\sqrt{4^2+3^2}$ $= 80$ $\therefore$ BE $= 40$ $\therefore$ AE $= \sqrt{96^2 - 40^2} = 8\sqrt{144 - 25} = 8\sqrt{119}$ $= 87 \cdot 28$. Area of ABCD $= \frac{1}{2}$BC . CD $+ \frac{1}{2}$AE . BD $= 32 \times 48 + 40$ $\times 87 \cdot 28 = 5027$ sq. yds.

74. PQ $= 5405$ yds.

75. Complete the rect. CBFH. Join HB, and produce it to meet DA in K. Draw KLM ∥ to ABF to meet CB, HF in L, M. BLMF is the reqd. fig. by II. 10.

76. If the duck enters the water at A, descends to B, and emerges at C. ABC is an isos. △. Draw BD perp. to AC. ABD is half an equilat. △ $\therefore$ AC $= 2$AD $= 2 \times 9\sqrt{3} = 31 \cdot 18$ feet. BD $=$ $\frac{1}{2}$AB $= 9$ feet.

77. Let ABC be the △, ∠B being a rt. ∠, ACDE the road. Draw AF, CG perp. to ED. △s AFE, CGD are isos. and rt. ∠d. $\therefore$ AE $= 11 \cdot 5 \times \sqrt{2}$ $\therefore$ area of ground $= \frac{1}{2}$BE . BD $= \frac{1}{2}(150 + 11 \cdot 5$ $\times \sqrt{2})^2 = 13821 \cdot 76$ sq. metres $\therefore$ cost $=$ £6911 to the nearest pound.

78. If A is the top of the tower, B the observer's eye, and BC perp. to the tower. ABC is half an equilat. △ $\therefore$ AC $=$ $200\sqrt{3} = 346 \cdot 4$ ft. $\therefore$ ht. of tower $= 351 \cdot 4$ feet.

79. Let ABCD be the sq., EF one of the ∥ lines cutting AB at E. Draw EM perp. to BD. BE $=$ EM$\sqrt{2} = \sqrt{2}$ cms. $\therefore$ AE $=$ $12 - \sqrt{2}$, △AEF $= \frac{1}{2}(12 - \sqrt{2})^2 = 56 \cdot 03$ sq. cms. Area of middle portion $= 144 - 112 \cdot 06 = 31 \cdot 94$ sq. cms.

80. The area is made up of a △ and four trapeziums. Total area $= \frac{3}{2} + \frac{5}{2} + \frac{6}{2} + \frac{8\frac{1}{2}}{2} + \frac{7\frac{1}{2}}{2} = 15$ sq. in.

81. If we suppose the vertical surface of the tower unwrapped until it lies in a plane, the rope will then run in a str. line along a diagonal of the rectangle thus formed. Length of rope $= \sqrt{48^2 + 20^2} = 52$ feet.

EXERCISES LXV. (*Continued.*)

4. Draw any regular hexagon, reduce it to a △, and following the construction in V. 26. obtain the line GM, so that GM^2 = the area of the hexagon.

On a str. line PQ, 2·36 in. long, describe a semicircle, and from PQ cut off PN equal to 1 in. Draw NR perp. to PQ to meet the circle in R. $PR^2 = PN \cdot PQ = 2{\cdot}36$.

Now proceed as in V. 26., making $AB : AB' = GM : PR$.

The proof is the same as in V. 26.

5. and **6.** Similar to 4, but starting with any regular pentagon and octagon respectively.

7. Use the method of V. 26. A, taking $n = 3$.

8. As in V. 26. A, we have to get Ab such that $\frac{Ab}{AB} = \frac{1}{\sqrt{5}}$.

Hence on FG (5 half-inches long) describe a semicircle, from FG cut off FH equal to 1 half-inch, and draw HK perp. to FG to meet the circumference in K. $FK^2 = FH \cdot FG = 5$. $\therefore$ $FK = \sqrt{5}$.

$\therefore \frac{FH}{FK} = \frac{1}{\sqrt{5}}$, and we then find A$b$ such that $\frac{Ab}{AB} = \frac{FH}{FK}$.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.

Cambridge Mathematical Series.

Crown 8vo.

ALDIS (W. S.). RIGID DYNAMICS, An Introductory Treatise on. By W. Steadman Aldis, M.A., Trinity College, Cambridge. 4s.

—— GEOMETRICAL OPTICS. An Elementary Treatise. 8th edition, revised. 4s.

BAKER (W. M.). ELEMENTARY DYNAMICS. By W. M. Baker, M.A., Headmaster of the Military and Civil Department at Cheltenham College. 4th edition, revised and enlarged. 4s. 6d. Key, 10s. 6d. net.

—— THE CALCULUS FOR BEGINNERS. 3s.

—— ALGEBRAIC GEOMETRY. A New Elementary Treatise on Analytical Conic Sections. 3rd edition. 6s. Part I. (The Straight Line and the Circle), 3rd ed., separately, 2s. 6d. Key, 7s. 6d. net.

BAKER AND BOURNE. PUBLIC SCHOOL ARITHMETIC. By W. M. Baker, M.A., Headmaster of the Military and Civil Department at Cheltenham College, and A. A. Bourne, M.A., late Head Mathematical Master at Cheltenham College. 2nd edition. 3s. 6d.; or with Answers, 4s. 6d. Answers separately, 1s. net.

—— STUDENT'S ARITHMETIC. Being a shortened form of the "Public School Arithmetic" (containing all of the Examples from that book). 4th edition. With or without Answers, 2s. 6d. Answers separately, 6d. net.

—— ARITHMETIC. Being the Public School Arithmetic in two Parts. With or without Answers, 2s. each.

—— EXAMPLES IN ARITHMETIC. 2s.

—— ELEMENTARY ALGEBRA. Complete, with or without Answers, 13th edition, thoroughly revised. 4s. 6d.

PART I. 19th edition. To Quadratic Equations, 2s. 6d.; or with Answers, 3s.

PART II. 7th edition. With or without Answers, 2s. 6d.

ANSWERS separately, 1s. net.

COMPLETE KEY, 3rd ed., 10s. net; or in Two Parts, 5s. net each.

—— EXAMPLES IN ALGEBRA. Extracted from the above. 5th Edition, revised, with or without Answers, 3s.; or Part I., without Answers, 1s. 6d., with Answers, 2s.; Part II., with or without Answers, 2s.

—— A SHORTER ALGEBRA. Suitable for Junior Oxford and Cambridge Local, and similar Examinations. 3rd edition. With or without Answers. 2s. 6d.

—— A NEW GEOMETRY. Uniform with "The Student's Arithmetic." 2nd edition. 2s. 6d. Also Books I.-III. separately, 1s. 6d.

—— ELEMENTARY GEOMETRY. A Text-book of Modern Practical and Theoretical Geometry. Complete, 8th edition, revised, 4s. 6d.

Also published in the following forms:

Book I., 11th edition, 1s.; Books I. and II., 10th edition, 1s. 6d.; Books I.-III., 13th edition, revised, 2s. 6d.; Books II. and III., 5th edition, 1s. 6d.; Book IV., 4th edition, 1s.; Books III. and IV., 12th edition, 1s. 6d.; Books II.-IV., 3rd edition, 2s. 6d.; Books I.-IV., 11th edition, 3s.; Book V., 1s. 6d.; Books IV. and V., 2nd edition, 2s.; Books IV.-VII., 6th edition, 3s.;

Books V.-VII., 4th edition, 2s. 6d.; Books VI. and VII., 3rd edition, 1s. 6d.

ANSWERS TO THE NUMERICAL AND MENSURATION EXAMPLES, 6d. net. COMPLETE KEY. 4th edition, revised. 6s. net.

BAKER AND BOURNE. ELEMENTARY GRAPHS. 5th ed. 6d. net.

BESANT (W. H.). CONIC SECTIONS TREATED GEOMETRICALLY. By W. H. Besant, Sc.D., F.R.S., Fellow of St. John's College, Cambridge. 11th edition. 4s. 6d.

SOLUTIONS TO THE EXAMPLES. 5s. net.

—— ELEMENTARY CONICS, being the first 8 chapters of the above. 2nd edition. 2s. 6d.

—— ELEMENTARY HYDROSTATICS. 21st edition. 4s. 6d.

SOLUTION TO THE EXAMPLES. 5s. net.

—— ROULETTES AND GLISSETTES, Notes on. 2nd edition, enlarged. 5s.

BORCHARDT AND PERROTT. A NEW TRIGONOMETRY FOR SCHOOLS. By W. G. Borchardt, M.A., Assistant Master at Cheltenham College, and the Rev. A. D. Perrott, M.A., Inspector of Schools, Diocese of Ely. 11th edition, with or without Answers. 4s. 6d. Also in 2 Parts, 2s. 6d. each. Answers separately, 6d. net.

COMPLETE KEY, 10s. net; or in two Parts, 5s. net each.

—— GEOMETRY FOR SCHOOLS. Complete, 4s. 6d. net.

Also published in the following forms:

Vol. I. Introductory and Experimental, Covering Stages I. and II. of the Board of Education Circular, No. 711, 2nd edition, 1s. Vol. II. Properties of Triangles and Parallelograms, 1s. 6d. Vol. III. Areas, 1s. Vols. I.-III. in one volume, 2s. 6d. Vol. IV. Circles, 1s. Vols. I.-IV. in one volume, 3s. Vol. V. Proportion, 1s. Books IV. and V. in one volume, 2s. Vols. I.-V. in one volume, 3s. 6d. Vol. VI. Solids, 1s. 6d. Vols. II.-VI. Cover Stage III. of the Board of Education Circular. COMPLETE KEY, 8s. 6d. net.

—— A SHORTER GEOMETRY. 2s. 6d. [*In preparation.*

—— A FIRST NUMERICAL TRIGONOMETRY. 2s. 6d.

—— A JUNIOR TRIGONOMETRY. Containing the matter of the "First Numerical Trigonometry," with supplementary chapters covering the ground of the Oxford and Cambridge Junior Locals and similar Examinations. With or without Answers, 3s. 6d.

CHARLES AND HEWITT. EXPERIMENTAL MECHANICS FOR SCHOOLS. By F. Charles, B.A., Senior Mathematical Master, Strand School, King's College, London, and W. H. Hewitt, B.A., B.Sc., A.R.C.S., Senior Science Master, Strand School, King's College, London. 3s. 6d.

DAVISON (C.). MATHEMATICAL PROBLEM PAPERS. By C. Davison, M.A., Sc.D., Mathematical Master at King Edward's School, Birmingham. 2s. 6d.

—— DIFFERENTIAL CALCULUS FOR COLLEGES AND SECONDARY SCHOOLS. [*In the Press.*

DEIGHTON (H.). EUCLID. Books I.-VI. and Part of Book XI. By Horace Deighton, M.A., formerly Scholar of Queen's College, Cambridge; Head Master of Harrison College, Barbados. Revised edition. 4s. 6d. Also in Parts. Book I., 1s. Books I. and II., 1s. 6d. Books I.-III., 2s. 6d. Books I.-IV., 3s. Books III. and IV., 1s. 6d. Books V.-XI., 2s. 6d.

DEIGHTON AND EMTAGE. INTRODUCTION TO EUCLID, including Euclid I., 1-26, with Explanations and numerous Easy Exercises. By Horace Deighton, M.A., and O. Emtage, B.A. 2nd edition, 1s. 6d.

DYER AND WHITCOMBE. ELEMENTARY TRIGONOMETRY. By J. M. Dyer, M.A., and the Rev. R. H. Whitcombe, M.A., Eton College. 3rd edition. 4s. 6d.

GALLATLY (W.). PHYSICS, Examples in Elementary. Comprising Statics, Dynamics, Hydrostatics, Heat, Light, Chemistry, Electricity, with Examination Papers. By W. Gallatly, M.A., Pembroke College, Cambridge, Assistant Examiner at London University. 4s.

GARNETT (W.). ELEMENTARY DYNAMICS, A Treatise on, for the use of Colleges and Schools. By William Garnett, M.A., D.C.L. 5th edition, revised. 6s.

—— HEAT, an Elementary Treatise on. 6th edition. 4s. 6d.

HATHORNTHWAITE (J. T.). ELEMENTARY ALGEBRA for the Entrance Examinations of the Indian Universities. By J. T. Hathornthwaite, M.A. 2s.

JESSOP AND CAUNT. THE ELEMENTS OF HYDROSTATICS. By C. M. Jessop, M.A., late Fellow of Clare College, Cambridge, Professor of Mathematics, Armstrong College, and G. W. Caunt, M.A., Lecturer in Mathematics, Armstrong College. 2s. 6d.

JESSOP AND HAVELOCK. ELEMENTARY MECHANICS. By C. M. Jessop, M.A., and T. H. Havelock, M.A., D.Sc., Fellow of St. John's College, Cambridge, Lecturer in Applied Mathematics, Armstrong College, Newcastle-on-Tyne. 3rd edition. 4s. 6d.

LODGE (A.). DIFFERENTIAL CALCULUS FOR BEGINNERS. By Alfred Lodge, M.A., Mathematical Master at Charterhouse, late Fereday Fellow of St. John's College, Oxford, and Professor of Pure Mathematics at Coopers Hill. With an Introduction by Sir Oliver Lodge, D.Sc., F.R.S., LL.D., Principal of the University of Birmingham. 4th edition, revised, with new sets of Miscellaneous Examples and Examination Papers. 4s. 6d.

—— INTEGRAL CALCULUS FOR BEGINNERS. 2nd ed. 4s. 6d.

McDOWELL (J.). EXERCISES ON EUCLID AND IN MODERN GEOMETRY, containing Applications of the Principles and Processes of Modern Pure Geometry. By J. McDowell, M.A., F.R.A.S. 4th edition. 6s.

MARSHALL AND TUCKEY. EXAMPLES IN PRACTICAL GEOMETRY AND MENSURATION. By J. W. Marshall, M.A., and C. O. Tuckey, M.A., Assistant Masters at Charterhouse. 3rd edition, with or without Answers, 1s. 6d.

MINCHIN (G. M.). THE STUDENT'S DYNAMICS. Comprising Statics and Kinetics. By G. M. Minchin, M.A., F.R.S., sometime Professor of Applied Mathematics at Coopers Hill. 2nd edition, revised. 3s. 6d.

PENDLEBURY (C.). NEW SCHOOL ARITHMETIC. By Charles Pendlebury, M.A., F.R.A.S., Senior Mathematical Master of St. Paul's School, late Scholar of St. John's College, Cambridge, assisted by F. E. Robinson, M.A. Complete, with or without Answers. 13th edition. 4s. 6d.

In Two Parts, with or without Answers, 2s. 6d. each. The Answers separately, 6d. net. Key to Part II., 8s. 6d. net.

—— NEW SCHOOL EXAMPLES IN ARITHMETIC. Extracted from the above. 8th edition, with or without Answers, 3s.; or in two Parts (without Answers), 1s. 6d. and 2s.

PENDLEBURY (C.). ARITHMETIC. With 8000 Examples. Complete, with or without Answers. 24th edition. 4s. 6d.

In Two Parts, with or without Answers, 2s. 6d. each. The Answers separately, 6d. net. Key to Part II. 6th edition. 7s. 6d. net.

Also a Colonial Edition, with or without Answers, 3s. 6d.

—— EXAMPLES IN ARITHMETIC. Extracted from the "Arithmetic," 17th edition. With or without Answers, 3s., or in Two Parts (without Answers), 1s. 6d. and 2s.

—— ELEMENTARY TRIGONOMETRY. 5th edition. 4s. 6d.

—— A SHORT COURSE OF ELEMENTARY PLANE TRIGONOMETRY. 3rd edition. 2s. 6d.

PENDLEBURY AND BEARD. COMMERCIAL ARITHMETIC. By C. Pendlebury, M.A., and W. S. Beard, F.R.G.S. 9th edition. Crown 8vo. 2s. 6d.; or Part I., 1s.; Part II., 1s. 6d.

PENDLEBURY AND TAIT. ARITHMETIC FOR INDIAN SCHOOLS. By C. Pendlebury, M.A., and J. S. Tait, M.A., B.Sc., Principal of Baroda College. 8th edition. 3s.

STAINER (W. J.). JUNIOR PRACTICAL MATHEMATICS. By W. J. Stainer, B.A., Headmaster, Municipal Secondary School, Brighton. 2s.; or with Answers, 2s. 6d. Also Part I. (consisting chiefly of Arithmetic and Algebra), 1s. 4d.; or with Answers, 1s. 6d. Part II. (Geometry and Mensuration), 1s. 4d.

STERN AND TOPHAM. PRACTICAL MATHEMATICS. By H. A Stern, M.A., late Open Mathematical Scholar of Corpus Christi College, Oxford, etc., and W. H. Topham, Science Master at Repton. New and revised ed. 6s.; or in 2 parts. Part I., 2s. 6d. Part II., 3s. 6d.

TAYLOR (C.). CONICS, The Elementary Geometry of. By C. Taylor, D.D., Master of St. John's College, Camb. 8th edition, revised. 5s.

TUCKEY (C. O.). EXAMPLES IN ARITHMETIC, with some Notes on Method. By C. O. Tuckey, M.A., Assistant Master at Charterhouse, late Scholar of Trinity College, Cambridge. 2nd edition, revised. With or without Answers, 3s.

—— EXAMPLES IN ALGEBRA. 5th edition, revised. With or without Answers, 3s. Appendix of Supplementary Examples. 6d. net.

UNWIN (P. W.). PRACTICAL SOLID GEOMETRY. By the Rev. P. W. Unwin, M.A., Assistant Master in the Military and Civil Department, Cheltenham College. 4s. 6d.

VYVYAN (T. G.). TRIGONOMETRY, Introduction to Plane. By the Rev. T. G. Vyvyan, M.A., formerly Fellow of Gonville and Caius College, Senior Math. Master at Charterhouse. 3rd ed. 3s. 6d.

—— ANALYTICAL GEOMETRY FOR BEGINNERS. Part I., The Straight Line and Circle. 3rd edition. 2s. 6d.

WALTON (W.). MECHANICS, A Collection of Problems in Elementary. By W. Walton, M.A., late Fellow of Trinity Hall. 2nd ed. 6s.

WHITWORTH (W. A.). CHOICE AND CHANCE. An Elementary Treatise on Permutations, Combinations, and Probability, with 640 Exercises. By W. A. Whitworth, M.A., late Fellow of St. John's College, Cambridge. 5th edition, revised. 7s. 6d.

—— DCC. EXERCISES: including Hints for the Solution of the first 700 Questions in Choice and Chance. 6s.

WILLIS (H. G.). CONIC SECTIONS, An Elementary Treatise on Geometrical. By H. G. Willis, M.A., Assistant Master of Manchester Grammar School. 5s.

LONDON: G. BELL & SONS, LTD.
YORK HOUSE, PORTUGAL STREET, W.C.